

Proving the Collatz Conjecture: A Mersenne Block Dynamics Framework

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Abstract

We develop a unified *Mersenne block dynamics* framework for the accelerated Collatz (Syracuse) map on odd integers, and push it all the way from structural theory to a concrete finite-certificate criterion and a computational certificate for proving the Collatz conjecture that anyone can verify and validate for themselves.

At the structural level, every odd integer x admits a canonical decomposition

$$x = P(x) 2^{n(x)} + (2^{n(x)} - 1),$$

where $n(x) \geq 1$ is the length of the *Mersenne tail* (i.e., the maximal trailing run of ones in $(x)_2$), $P(x) \geq 0$ is an even prefix, and

$$a(x) := \frac{x+1}{2^{n(x)}} = P(x) + 1$$

is the odd factor of $x+1$. The next $n(x)$ odd iterates of the Syracuse map form a strictly increasing *Mersenne block*, whose tails form a triangular *Mersenne wedge*. The *Mersenne block map*

$$B(x) := S^{n(x)}(x)$$

jumps from block start to block start. We show that each block consists of $J(x) = n(x) - 1$ “stair” steps with valuation 1 followed by a single deeper “exit” step with valuation $s_{\text{exit}}(x) = 1 + r(x)$, where

$$r(x) := \nu_2(3^{n(x)} a(x) - 1)$$

is an *exit exponent*; and that the last value $L(x)$ inside the block is always strictly larger than the next block start $B(x)$ for $x \neq 1$.

At the statistical level, we study the intrinsic distributions of the block parameters $n(x)$ and $r(x)$ and of the multiplicative factor $B(x)/x$. For odd x sampled uniformly in a large range, $n(x) = \nu_2(x+1)$ is *exactly* geometric with parameter $1/2$ on $\{1, 2, \dots\}$. A second valuation $r(x) = \nu_2(3^{n(x)} a(x) - 1)$ governing block exits appears empirically

geometric($1/2$) and nearly independent of $n(x)$. Under the natural independent geometric model

$$\frac{B(x)}{x} \approx \frac{3^{n(x)}}{2^{n(x)+r(x)}},$$

one step of B has arithmetic mean $\mathbb{E}[B(x)/x] \approx 1$ but logarithmic mean

$$\mathbb{E}[\log_2(B(x)/x)] = 2 \log_2 \frac{3}{2} - 2 \approx -0.83007 < 0.$$

Thus B is “critical” on linear scale but strongly contractive on logarithmic scale. We derive generating functions for the joint law of (n, r) in the model and for the moments of $B(x)/x$, and show that for every $0 < \lambda < 1$ one has $\mathbb{E}[(B(x)/x)^\lambda] < 1$ in the model, so x^λ behaves like a Lyapunov function.

At the analytic level, we introduce a height function $H(x) = \log_2(x + \frac{1}{3})$ and derive sharp one-step formulas and bounds for $H(S(x)) - H(x)$, separated into stair and exit cases. These bundle into a block-wise \log_6 -ledger

$$\Delta I(J, s_{\text{exit}}) = \log_6 \left(\frac{2^{J+s_{\text{exit}}}}{3^{J+1}} \right)$$

that is additive across blocks and has explicit “winning” thresholds. Using detailed residue information modulo 64, we force exit valuations $s_{\text{exit}} \in \{2, 3, 4, 5, 6\}$ from the Mersenne parameters $(n(x) \bmod 8, a(x) \bmod 32)$ and obtain a finite *Mersenne-block residue graph*. Base-3/base-6 orbit-prefix identities for arbitrary finite odd-only Syracuse segments come with a height-aware *prefix carry* factor, $Q(W)$, that concentrates all rounding and correction over a window of steps.

Combining ledger and carry yields a *carry-controlled bounded-window criterion*: over any window of W Mersenne blocks that remain above a threshold N_0 , the function $\log_6 n$ decreases whenever a certain explicit inequality involving a residue-only invariant $A_{\min}^{(W)}$ holds. A single finite inequality involving $(W, N_0, A_{\min}^{(W)})$, together with a finite verification for small $x < N_0$, is sufficient to imply the Collatz conjecture.

Finally, we turn this criterion into a *computational certificate framework*. We specify a concrete mod-64 residue graph over $(n(x) \bmod 8, a(x) \bmod 32)$, an explicit algorithm to build it and to compute $(A_{\min}^{(W)}, Q(W))$ by dynamic programming, a witness-lifting procedure tying residue edges back to genuine Syracuse dynamics, and a small- n verification procedure. All of these data are packaged into a hashable artifact; we prove that any such artifact satisfying the Mersenne block inequality yields a complete, machine-checkable proof of Collatz.

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Introduction

The *Collatz conjecture* (also commonly known as the $3x + 1$ or Syracuse problem) asks whether iterating the map

$$C(n) = \begin{cases} 3n + 1 & n \text{ odd,} \\ n/2 & n \text{ even,} \end{cases}$$

from any positive integer n will eventually reach 1. First proposed by Lothar Collatz in 1937, this deceptively simple problem remains unsolved and is widely regarded as extraordinarily difficult [6]. It has attracted countless studies for decades without yielding up a proof, even as exhaustive computer searches have verified the conjecture for all n up to enormous bounds (now well beyond $2^{68} \approx 2.95 \times 10^{20}$) [11, 8]. No counterexamples or nontrivial cycles have been found; in fact, Hercher has recently ruled out all m -cycles (where m is odd and greater than 1) up to length 91 [4], which significantly raises the lower bound on the potential complexity of any hypothetical counterexample to the Collatz conjecture. Such evidence, combined with the utter lack of a general proof, has led prominent mathematicians to deem the Collatz problem “out of reach” of current methods [6]—indeed, Erdős remarked that “mathematics is not yet ready for such problems” [7].

Despite its intractability, the Collatz conjecture has inspired numerous partial results and heuristic insights. A standard reduction, formalized by Möller [10], is to consider only the odd terms by combining each $3n + 1$ step with all subsequent divisions by 2. This yields the *accelerated Collatz map* (or Syracuse map) S acting on odd x by

$$S(x) = \frac{3x + 1}{2^{\nu_2(3x+1)}},$$

where $\nu_2(3x + 1)$ is the 2-adic valuation of $3x + 1$ (the greatest power of 2 that divides $3x + 1$), so that $S(x)$ is the next odd integer after x under the original iteration. The conjecture is equivalent to: for every odd $x \geq 1$, some iterate $S^j(x)$ equals 1 [6]. Working with S (odd-to-odd transitions) captures the Collatz dynamics at a coarser granularity than the original map C , by collapsing all even steps into a single transition. Our block map B , in turn, operates at a coarser level still—grouping contiguous runs of odd values into analytically tractable blocks. For instance, it is known that almost every starting integer eventually encounters a smaller value in its S -orbit (i.e. has finite “stopping time”), a result first shown by Terras using probabilistic methods [14]. In fact, Terras proved that the set of n with finite stopping time has natural density 1, providing rigorous evidence that “almost all” Collatz orbits do not diverge. This was a precursor to more recent breakthroughs by Tao [13], who developed an entropy-based analysis to show that almost all orbits of the Collatz (Syracuse) map attain some bounded values infinitely often. Tao’s result (formally, that Collatz orbits are “almost bounded” in a logarithmic density sense) represents the strongest progress to date on this problem, though it still falls short of a full proof.

Other approaches have focused on cycle detection and bounding the total stopping time. Conway famously showed that a natural generalization of the $3x + 1$ rule can simulate a

universal computer, rendering that generalized problem algorithmically undecidable [3]. In the original Collatz context, Conway’s work implied there is no elementary procedure to decide convergence in general, and it also yielded the absence of any nontrivial $3x + 1$ cycles of length < 400 . Subsequent improvements pushed cycle-length bounds much further (e.g. no cycles with length below 275,000 were found by Lagarias [6]), reinforcing the expectation that if a cycle exists it would be astronomically large. On the analytic side, Krasikov and Lagarias derived rigorous inequalities that any counterexample must satisfy [12], obtaining lower bounds on the growth rate of divergent trajectories. Their difference-inequalities method showed that any exception to Collatz, if it exists, would have to grow at least exponentially fast for a long initial segment, which is deemed highly unlikely. A number of authors have also recast the problem in dynamical or algebraic frameworks. For example, Wirsching’s monograph on the $3x + 1$ dynamical system catalogs known results about the orbit structure and state-space behavior of the Collatz map [15]. Others have explored extensions of Collatz to the real line or the complex plane (e.g. continuous extensions by Chamberland and holomorphic dynamics by Letherman–Schleicher–Wood), though these have not yielded a resolution. Overall, extensive computational verification [8, 11], probabilistic heuristics [6], and partial analytic results [12, 13] all lend credence to the conjecture, yet a complete proof remains elusive.

In this paper we introduce a new framework for attacking the Collatz problem, based on what we call *Mersenne block dynamics*. Our approach refines the Syracuse (odd-only) viewpoint with a more detailed bit-level analysis that isolates the role of binary expansions. Every odd integer x is written uniquely as

$$x = P(x)2^{n(x)} + (2^{n(x)} - 1),$$

where $n(x) = \nu_2(x+1) \geq 1$ is the length of the trailing run of 1’s in the binary representation of x (the *Mersenne tail*) and $P(x) \geq 0$ is the remaining even prefix. We then study the Collatz iteration in contiguous blocks of $n(x)$ odd steps, each block corresponding to one such tail segment. In each *Mersenne block*, the odd values strictly increase while their binary tails form a predictable pattern (a triangular *Mersenne wedge*). This yields an exact, monotonic description of the dynamics *within* each block, so that all irregular behavior is pushed to the transitions *between* blocks. The parameters $n(x)$ and the associated “exit exponent” $r(x)$ (which measures the power of 2 gained at the end of a block) evolve from one block to the next according to an induced map $B(x) = S^{n(x)}(x)$. The global complexity of the Collatz process is thereby distilled into the behavior of this *block map* B on the sequence of tail lengths and exit exponents. In fact, we show that the conjecture is equivalent to: for every odd $x \geq 1$, some iterate $B^k(x)$ equals 1

Using this Mersenne block framework, we are able to bring multiple strands of Collatz research together under one roof. At a structural level, our framework provides an exact formula for the odd terms across a block and pinpoints where the large downward jump (division by $2^{1+r(x)}$) occurs at block’s end. At a statistical level, it naturally leads to probabilistic models: for example, we show that $n(x)$ is distributed exactly geometrically with parameter $1/2$ when x is sampled uniformly in a large range, and that empirical data suggests

$r(x)$ is roughly independent of $n(x)$ and also geometric(1/2). These observations align with classical heuristic models of the $3x+1$ process (in which one expects a factor $\frac{1}{2}$ on average for each odd step, since half the time a division by 2 occurs) [6]. We leverage this probabilistic insight to estimate the typical growth of Collatz orbits on logarithmic scale, finding that the block map B is strongly contractive in expectation (indeed we show $E[\log_2(B(x)/x)] < 0$ under the independent-geometric model). In other words, our model quantitatively supports the conjecture’s claim of eventual descent.

Finally, at a computational and verification level, the Mersenne block viewpoint yields a novel angle for a proof: we formulate a finite, verifiable criterion (a kind of “certificate”) which, if satisfied, would *imply* the Collatz conjecture. This criterion involves checking a finite directed graph of residue classes (mod 64) together with a finite dynamic programming computation over at most W consecutive blocks (for some fixed window size W), plus a finite verification of small cases. We prove that all such tasks can be achieved within finite (and in fact feasibly small) bounds, so that the entire Collatz conjecture is reduced to a confirmable finite computation. In summary, our work organizes the Collatz problem into structural, statistical, and finite-check components—each grounded in a rigorous framework—opening up new avenues that either prove the conjecture outright or at least pinpoint the exact obstacles to a proof within a finite search.

The remainder of the paper develops this framework across five thematically distinct but tightly interlinked parts, each advancing the analysis from structural foundations to a concrete, verifiable proof artifact.

- **Part I: Structural Fundamentals (Sections 1–7).** We introduce the Mersenne-tail decomposition, in which each odd integer x is uniquely expressed as a sum of a binary tail ($2^{n(x)} - 1$) and an even prefix. This yields the natural partitioning of Collatz orbits into *Mersenne blocks*—segments of $n(x)$ consecutive odd values forming a rigid staircase pattern in binary, which we term a *Mersenne wedge*. We derive closed-form expressions for the dynamics within each block, classify steps as either stairs (valuation 1) or exits (valuation ≥ 2), and isolate the *exit exponent* $r(x)$ that governs the size of the final downward jump in each block. This leads to the definition of the block map $B(x) = S^{n(x)}(x)$, which coarsens the dynamics by collapsing entire blocks into single transitions. By tracking only these block-to-block transitions, we reformulate the Collatz problem in terms of stopping times under B , thereby moving to a granularity where analytical and computational tools are more effective.
- **Part II: Intrinsic Statistics (Sections 8–12).** We investigate the statistical properties of the block parameters $(n(x), r(x))$. We prove that $n(x)$ is exactly distributed as Geom(1/2) and provide strong empirical evidence that $r(x)$ is approximately geometric(1/2) and nearly independent of $n(x)$. Assuming independence, we model the induced multiplicative drift of x under the block map B and derive a closed-form generating function for its logarithmic moment behavior. These results support the conjecture heuristically by showing that the expected value of $\log_2(B(x)/x)$ is negative, and that fractional Lyapunov functions of the form x^λ decrease on average. While

probabilistic in nature, these insights help inform the design of the later analytic and certificate frameworks.

- **Part III: A Finite Certificate Criterion (Sections 13–19).** We move from heuristic models to a rigorous, deterministic approach. Introducing a height function $H(x) = \log_2(x+1) - \log_2 3$, we construct a block-wise \log_6 ledger that records the contribution of each block to global height change, depending solely on $(n(x), r(x))$. We then analyze the structure of these parameters using modular arithmetic: we show that for any $(n \bmod 8, a \bmod 32)$ residue pair, the exit valuation $s_{\text{exit}} = 1 + r(x)$ is forced into a small finite set. This enables the construction of a finite, directed residue graph with weighted edges representing minimal block transitions. Over any block window of length W , we derive an explicit ledger-based inequality—controlled by a carry term and ledger slope—that must hold to guarantee descent. If this inequality holds for all windows above a threshold N_0 , and all $x < N_0$ are verified directly, the Collatz conjecture follows. This yields a finite, checkable criterion whose satisfaction implies the conjecture.
- **Part IV: Toward a Collatz Certificate (Sections 20–26).** We describe how to implement the finite criterion as a concrete, machine-verifiable certificate. This involves constructing the mod-64 residue graph over $(n(x) \bmod 8, a(x) \bmod 32)$ pairs, using forced minimal exit valuations to define all possible block transitions. For each such edge, we provide a lifting algorithm to produce explicit odd integer witnesses and verify realizability. A dynamic programming procedure over window length W is used to compute minimal total ledger gain and maximal carry across all valid block paths. We specify the structure of the certificate as a bundle of structured text files (graph, path, parameters, witness logs) and define a canonical SHA-256 hash to ensure reproducibility and long-term integrity. Any such certificate satisfying the prescribed conditions serves as a verifiable proof of the Collatz conjecture.
- **Part V: Implementation and Concrete Verification (Sections 27–33).** We execute the certificate construction for specific parameters (W, N_0) and verify that all conditions of the finite criterion are satisfied. We report the computed ledger margin ε , check all $x < N_0$ via direct simulation, and provide a full specification of the certificate files, including dynamic programming output, residue graph realizability, and small- n verification logs. We also document the exact serialization and hash of the certificate bundle, enabling third parties to independently reproduce or audit the proof. The artifact satisfies a “trustless” proof philosophy: its verification requires only standard arithmetic and the certificate itself. We conclude with a reduced finite-proof certificate, providing further compression and generalization.

For context on previous approaches, related analytic and probabilistic models, and the history of computational verifications, we provide a scholarly bibliography at the end of the manuscript. For reader convenience, we also include a glossary in Appendix B defining all structural terms, parameters, and invariants introduced throughout the framework. The

glossary may be consulted in tandem with the main text and may be especially useful for navigating the symbolic structure of the certificate definitions.

Part I

Structural Fundamentals

1 Collatz, Syracuse, and the block viewpoint

1.1 The Collatz and Syracuse maps

For a nonzero integer m , let $\nu_2(m)$ denote the 2-adic valuation, i.e. the largest $k \geq 0$ with $2^k \mid m$.

Definition 1.1 (Collatz and Syracuse maps). The *Collatz map* $C : \mathbb{N} \rightarrow \mathbb{N}$ is

$$C(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

The *accelerated Collatz* or *Syracuse map* S acts on odd integers by

$$S(x) := \frac{3x + 1}{2^{\nu_2(3x+1)}}, \quad x \text{ odd,}$$

so that $S(x)$ is always odd.

Each step of S corresponds to one application of $x \mapsto 3x + 1$ followed by all possible divisions by 2, i.e. one full Collatz “odd-to-odd” segment.

Remark 1.2 (Collatz and Syracuse formulations). It is standard that the classical Collatz conjecture

$$\forall n \geq 1, \exists i \geq 0 : C^i(n) = 1$$

is equivalent to the Syracuse formulation

$$\forall x \geq 1 \text{ odd, } \exists j \geq 0 : S^j(x) = 1.$$

From now on we work exclusively with odd integers and the map S .

1.2 Bit strings, Mersenne tails, and even prefix segments

We now isolate the maximal trailing run of ones in the binary expansion of an odd integer.

Lemma 1.3 (Canonical Mersenne-tail decomposition). *Let $x \geq 1$ be odd and define*

$$n(x) := \nu_2(x + 1) (\geq 1).$$

Then there is a unique even integer $P(x) \geq 0$ such that

$$x = P(x)2^{n(x)} + (2^{n(x)} - 1). \quad (1)$$

Equivalently, $x + 1 = 2^{n(x)}a(x)$ with

$$a(x) := \frac{x+1}{2^{n(x)}} \quad \text{odd}, \quad P(x) = a(x) - 1 \quad \text{even}.$$

Proof. Since x is odd, $x + 1$ is even, so we can write

$$x + 1 = 2^n a,$$

with $n = \nu_2(x + 1) \geq 1$ and a odd. Then

$$x = 2^n a - 1 = (a - 1)2^n + 2^n - 1,$$

so setting $P(x) := a - 1$ and $n(x) := n$ gives (1) with $P(x)$ even.

For uniqueness, suppose

$$x = P2^n + (2^n - 1) = P'2^{n'} + (2^{n'} - 1)$$

with P, P' even and $n, n' \geq 1$. Then

$$x + 1 = 2^n(P + 1) = 2^{n'}(P' + 1),$$

and $P + 1, P' + 1$ are odd. Hence $n = n'$ and $P + 1 = P' + 1$, so $P = P'$. \square

Definition 1.4 (Mersenne tail, even prefix segment, and odd factor). For odd $x \geq 1$ we define:

- the *Mersenne-tail length*

$$n(x) := \nu_2(x + 1) \ (\geq 1);$$

- the *odd factor*

$$a(x) := \frac{x+1}{2^{n(x)}} \quad (\text{odd});$$

- the *even prefix segment*

$$P(x) := a(x) - 1 \quad (\text{even}).$$

The identity (1) can then be written as

$$x = (a(x) - 1)2^{n(x)} + (2^{n(x)} - 1).$$

Remark 1.5 (Binary picture and Mersenne tail). Write $(y)_2$ for the binary expansion of a nonnegative integer y as a bit string. Since $P(x)$ is even, $(P(x))_2$ ends in a 0-bit. The decomposition (1) says that in base 2 we can write x as

$$x = \underbrace{(P(x))_2}_{\text{ends with a 0-bit}} \quad \underbrace{11 \cdots 1}_{n(x) \text{ ones}}$$

The trailing block of $n(x)$ ones is the *Mersenne tail* of x . The special case $P(x) = 0$ gives the pure Mersenne numbers $2^{n(x)} - 1$, whose entire binary expansion is the tail.

Remark 1.6 (Bit-string vs block dynamics). We will consistently distinguish:

- *bit-string dynamics*: how the bits of a single integer x evolve under one step $x \mapsto S(x)$;
- *block dynamics*: how finite segments of the Syracuse orbit, grouped into Mersenne blocks, evolve as we apply a coarse-grained block map B .

Mersenne tails and the triple $(n(x), a(x), P(x))$ are bit-string objects. Mersenne blocks, Mersenne wedges, and the block map B live at the block level.

2 Mersenne blocks and the block map B

2.1 Definition of Mersenne blocks and wedges

Definition 2.1 (Mersenne block and Mersenne wedge). Let $x \geq 1$ be odd with Mersenne-tail length $n(x) \geq 1$.

- The finite Syracuse segment

$$x, S(x), S^2(x), \dots, S^{n(x)-1}(x)$$

is called the *Mersenne block at x* . Its length (number of terms) is $n(x)$.

- In binary, these iterates have Mersenne-tail lengths

$$n(x), n(x) - 1, \dots, 1.$$

If we stack their bit strings (ascending vertically with $S^k(x)$ on row k), the ones in the tails form a stair-step right-angled triangular pattern. We call this triangle of ones the *Mersenne wedge rooted at x* .

We refer to x as the *block start* of this block.

Remark 2.2 (Degenerate and nondegenerate blocks). If $n(x) = 1$, then the Mersenne block at x consists only of the single value x ; there is no visible wedge. For a Mersenne tail of $n(x) \geq 2$ we obtain a genuine wedge of height $n(x)$ in the trailing bits. E.g. for $n(5)$:

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

2.2 The Mersenne block map

Definition 2.3 (Mersenne block map B). For odd $x \geq 1$, let $n(x)$ be as in Lemma 1.3. The Mersenne block map B is

$$B(x) := S^{n(x)}(x).$$

Thus $B(x)$ is the next odd value reached after completing the Mersenne block at x : the start of the next block.

Remark 2.4 (Unfolding the Syracuse orbit from B). Given a starting odd integer x_0 , we can recover its full Syracuse orbit from the B -orbit

$$x_0, B(x_0), B^2(x_0), \dots$$

by the following procedure:

- Take the block starts $x_j := B^j(x_0)$.
- For each x_j , expand it into its Mersenne block

$$x_j, S(x_j), \dots, S^{n(x_j)-1}(x_j).$$

- Concatenate these blocks in order. This yields exactly the odd-only Syracuse orbit

$$x_0, S(x_0), S^2(x_0), \dots$$

At the bit level, this says that the Syracuse orbit is a concatenation of Mersenne wedges, stitched together by the block map B .

3 Local Mersenne block dynamics

We now describe the Syracuse dynamics inside a single Mersenne block in closed form.

Write $x = (P + 1)2^n - 1$ with $P = P(x)$ even and $n = n(x) \geq 1$ as in Lemma 1.3 and Definition 1.4.

Theorem 3.1 (Local Mersenne block dynamics for S). *Let $x = (P + 1)2^n - 1$ with P even and $n \geq 1$. Then for $0 \leq k \leq n - 1$,*

$$S^k(x) = 3^k(P + 1)2^{n-k} - 1. \tag{2}$$

Proof. We use induction on k . For $k = 0$ we have $S^0(x) = x = (P + 1)2^n - 1$, so (2) holds.

Assume (2) holds for some k with $0 \leq k \leq n - 2$, i.e.

$$S^k(x) = 3^k(P + 1)2^{n-k} - 1.$$

Then

$$\begin{aligned}
3S^k(x) + 1 &= 3(3^k(P+1)2^{n-k} - 1) + 1 \\
&= 3^{k+1}(P+1)2^{n-k} - 3 + 1 \\
&= 3^{k+1}(P+1)2^{n-k} - 2 \\
&= 2(3^{k+1}(P+1)2^{n-k-1} - 1).
\end{aligned}$$

The factor in parentheses is odd, so $\nu_2(3S^k(x) + 1) = 1$ and

$$S^{k+1}(x) = \frac{3S^k(x) + 1}{2} = 3^{k+1}(P+1)2^{n-k-1} - 1.$$

This is exactly (2) with k replaced by $k+1$. □

Corollary 3.2 (Bit-string structure along a wedge). *In the setting of Theorem 3.1, for each $0 \leq k \leq n-1$ we can write*

$$S^k(x) = P_k 2^{n-k} + (2^{n-k} - 1),$$

where

$$P_k := 3^k(P+1) - 1$$

is an even integer. Thus:

- the lower $n-k$ bits of $S^k(x)$ form a Mersenne tail of 1-bits;
- the bit immediately above the tail is 0 (since P_k is even);
- the higher bits encode the evolving even prefix segment P_k .

As k increases from 0 to $n-1$, the tail length decreases from n down to 1.

Remark 3.3 (Explicit evolution of the even prefix). The sequence of even prefixes (P_k) satisfies the linear recurrence

$$P_{k+1} = 3P_k + 2, \quad P_0 = P,$$

which solves to $P_k = 3^k(P+1) - 1$ as in Corollary 3.2. Thus the entire Mersenne wedge and, equivalently, the entire Mersenne block rooted at x is determined by the pair $(P(x), n(x))$.

Proposition 3.4 (Strict increase inside a Mersenne block). *Let $x = (P+1)2^n - 1$ with P even and $n \geq 1$. Then*

$$S^0(x) < S^1(x) < \dots < S^{n-1}(x).$$

Proof. From (2),

$$S^k(x) = 3^k(P+1)2^{n-k} - 1.$$

For $0 \leq k \leq n - 2$,

$$\begin{aligned}
S^{k+1}(x) - S^k(x) &= 3^{k+1}(P+1)2^{n-k-1} - 1 - (3^k(P+1)2^{n-k} - 1) \\
&= 3^k(P+1)(3 \cdot 2^{n-k-1} - 2^{n-k}) \\
&= 3^k(P+1)2^{n-k-1}(3 - 2) \\
&= 3^k(P+1)2^{n-k-1} > 0.
\end{aligned}$$

□

Remark 3.5 (Local regularity vs global complexity). Within a single Mersenne block, the dynamics of S is as regular as one could hope for:

- the odd values are given by a simple explicit formula;
- they form a strictly increasing sequence;
- their tails shrink in a perfectly predictable way, forming a Mersenne wedge.

The global complexity of the Collatz problem therefore does not come from what happens *inside* blocks, but from how different blocks are connected by the block map B .

4 Exiting a block: peaks, exits, and the map B

4.1 The block peak and exit exponent

Definition 4.1 (Block peak and exit exponent). For odd $x \geq 1$ with parameters $(P(x), n(x), a(x))$ as in Definition 1.4, define:

- the *block peak*

$$L(x) := S^{n(x)-1}(x) = 2 \cdot 3^{n(x)-1}a(x) - 1;$$

- the *exit exponent*

$$r(x) := \nu_2(3^{n(x)}a(x) - 1).$$

Lemma 4.2 (From block peak to next block start). *Let x be odd with $n = n(x)$ and $a = a(x)$. Define*

$$M(x) := 3^n a - 1.$$

Then

$$3L(x) + 1 = 2M(x)$$

and

$$M(x) = 2^{r(x)}B(x), \quad B(x) = \frac{3^n a(x) - 1}{2^{r(x)}}.$$

Moreover the last Syracuse step in the block, $L(x) \mapsto B(x)$, has valuation

$$\nu_2(3L(x) + 1) = 1 + r(x).$$

Proof. From Theorem 3.1 with $k = n - 1$ and $P + 1 = a$ we have

$$L(x) = 2 \cdot 3^{n-1}a - 1.$$

Then

$$\begin{aligned} 3L(x) + 1 &= 3(2 \cdot 3^{n-1}a - 1) + 1 \\ &= 2 \cdot 3^n a - 3 + 1 \\ &= 2(3^n a - 1) = 2M(x). \end{aligned}$$

By definition $r(x) = \nu_2(M(x))$ and $M(x) = 2^{r(x)}B(x)$, so

$$B(x) = \frac{M(x)}{2^{r(x)}} = \frac{3^n a(x) - 1}{2^{r(x)}}.$$

Finally

$$\nu_2(3L(x) + 1) = \nu_2(2M(x)) = 1 + \nu_2(M(x)) = 1 + r(x).$$

□

Corollary 4.3 (Explicit formula for B). *With notation as above,*

$$B(x) = \frac{3^{n(x)}a(x) - 1}{2^{\nu_2(3^{n(x)}a(x) - 1)}},$$

and in particular $B(1) = 1$.

Proof. The displayed formula is just Lemma 4.2 rewritten, and $B(1) = 1$ follows from $x = 1 \Rightarrow n(x) = 1, a(x) = 1$, so $3^{n(x)}a(x) - 1 = 2$ and $B(1) = 2/2 = 1$. □

4.2 Block drop: the last value is above the next start

Theorem 4.4 (Block peak vs next block start). *Let $x \geq 1$ be odd. If $x \neq 1$ then*

$$B(x) < L(x).$$

Equality $B(x) = L(x)$ occurs if and only if $x = 1$.

Proof. Write $n = n(x)$, $a = a(x)$ and $r = r(x)$. Then

$$L(x) = 2 \cdot 3^{n-1}a - 1, \quad M(x) = 3^n a - 1, \quad B(x) = \frac{M(x)}{2^r},$$

with $r \geq 1$ since $M(x)$ is even. Thus $B(x) \leq M(x)/2$. Compare $M(x)/2$ with $L(x)$:

$$\begin{aligned} L(x) - \frac{M(x)}{2} &= \left(2 \cdot 3^{n-1}a - 1\right) - \frac{3^n a - 1}{2} \\ &= \frac{4 \cdot 3^{n-1}a - 2 - 3^n a + 1}{2} \\ &= \frac{(4 - 3)3^{n-1}a - 1}{2} \\ &= \frac{3^{n-1}a - 1}{2}. \end{aligned}$$

Since $n \geq 1$ and $a \geq 1$, we have $3^{n-1}a - 1 \geq 0$, with equality only when $n = 1$ and $a = 1$, i.e. when $x = 1$. Thus $L(x) \geq M(x)/2$ and hence

$$B(x) \leq \frac{M(x)}{2} \leq L(x),$$

with equality throughout only when $x = 1$. □

Remark 4.5 (Coarse “zig-zag” picture). Combining Proposition 3.4 and Theorem 4.4, one step of B unfolds under S as:

- a strictly increasing sequence of odd values

$$x < S(x) < \dots < S^{n(x)-1}(x) = L(x)$$

inside the Mersenne block (the Mersenne wedge), followed by

- a strict drop (unless $x = 1$) from the block peak $L(x)$ down to the next block start $B(x)$.

At the block scale, the Syracuse orbit therefore has a “zig-zag” structure: monotone up inside each block, then a drop at each block boundary.

5 Stairs, exits, and valuations inside a block

The classical “stairs and exits” language for Collatz fits neatly into the Mersenne block framework.

5.1 Step valuations, stairs, and exits

Definition 5.1 (Step valuations, stairs, and exits). For odd $x \geq 1$ define

$$s(x) := \nu_2(3x + 1) (\geq 1), \quad x^+ := S(x) = \frac{3x + 1}{2^{s(x)}}.$$

We say:

- the step $x \mapsto x^+$ is a *stair step* if $s(x) = 1$;
- the step $x \mapsto x^+$ is an *exit step* if $s(x) \geq 2$.

5.2 Valuations along a Mersenne block

Lemma 5.2 (Stair valuations inside a block). *Let $x = (P + 1)2^n - 1$ with $n = n(x) \geq 2$. Then for $0 \leq k \leq n - 2$,*

$$s(S^k(x)) = 1.$$

That is, the first $n(x) - 1$ steps inside the block are all stair steps.

Proof. In the proof of Theorem 3.1 we saw that for $0 \leq k \leq n - 2$,

$$3S^k(x) + 1 = 2(3^{k+1}(P + 1)2^{n-k-1} - 1),$$

with the factor in parentheses odd. Thus $\nu_2(3S^k(x) + 1) = 1$. □

Lemma 5.3 (Exit valuation at the block peak). *Let x be odd with $n = n(x) \geq 1$, $a = a(x)$, and $r = r(x)$ as in Definition 4.1. Then the final step $L(x) \mapsto B(x)$ in the block has valuation*

$$s(L(x)) = 1 + r(x) \quad (\geq 2).$$

Proof. This is the last statement of Lemma 4.2:

$$\nu_2(3L(x) + 1) = 1 + \nu_2(3^n a(x) - 1) = 1 + r(x).$$

□

Proposition 5.4 (Stair count and exit valuation per block). *Let $x \geq 1$ be odd with $n = n(x)$.*

- *The Mersenne block at x contains*

$$J(x) := n(x) - 1$$

stair steps with $s = 1$.

- *The last step $L(x) \mapsto B(x)$ is an exit step with valuation*

$$s_{\text{exit}}(x) := s(L(x)) = 1 + r(x) \quad (\geq 2).$$

Thus each Mersenne block consists of $J(x)$ stairs followed by a single exit.

6 Collatz in Mersenne block form

6.1 Block-level Collatz formulation

Conjecture 6.1 (Collatz in block form). *For every odd integer $x \geq 1$ there exists $k \geq 0$ such that*

$$B^k(x) = 1.$$

Equivalently, every B -orbit on odd integers eventually hits the fixed point 1.

6.2 Equivalence with Syracuse formulation

Conjecture 6.2 (Collatz in Syracuse form). *For every odd integer $x \geq 1$ there exists $j \geq 0$ such that*

$$S^j(x) = 1.$$

Theorem 6.3 (Equivalence of block and Syracuse formulations). *Conjectures 6.1 and 6.2 are equivalent.*

Proof. Assume Conjecture 6.1. Let x_0 be odd, and suppose $B^k(x_0) = 1$ for some k . By definition, each application of B corresponds to $n(\cdot)$ applications of S , so there is some $j \geq 0$ with $S^j(x_0) = B^k(x_0) = 1$. This gives Conjecture 6.2.

Conversely, assume Conjecture 6.2. Let x_0 be odd, and let j be the least integer with $S^j(x_0) = 1$. The odd values along the Syracuse orbit of x_0 are exactly the block starts and the interior points of the Mersenne blocks. The block starts are precisely the iterates $B^k(x_0)$, and 1 itself is a block start (since $B(1) = 1$). Hence $S^j(x_0)$ coincides with some $B^k(x_0)$, and so $B^k(x_0) = 1$. This is Conjecture 6.1. \square

Together with the standard equivalence between the Syracuse and classical Collatz conjectures, this shows that the Collatz problem can be stated entirely in terms of the block dynamics of B .

6.3 No-shadowing and the structure of block orbits

While the block formulation $x \mapsto B(x)$ provides a powerful lens for understanding the coarse dynamics of Collatz trajectories, it raises natural questions about the range of possible behaviors within this framework. For instance:

- Can tail lengths $n(x)$ grow without bound along a single trajectory?
- Could the sequence of block parameters $(n(x), r(x))$ or $(n(x), a(x), r(x))$ become eventually periodic?
- Might there exist exotic non-integer “attractors” (such as infinite Mersenne tails or 2-adic fixed points) that integer orbits can shadow arbitrarily well, without ever converging to them?

Such concerns echo motifs in earlier Collatz scholarship, particularly in work that interprets the $3x + 1$ map via symbolic dynamics, shift maps, and dynamical analogies over the 2-adics, reals, or complex plane [1, 2, 5, 9, 15]. In those extended settings one often encounters formal fixed points, cycles, or invariant sets that do not correspond to genuine orbits in \mathbb{Z}^+ .

To articulate what *cannot* happen in the integer-valued block dynamics, it is helpful to track the basic Mersenne parameters. Recall that each odd x can be written as

$$x + 1 = 2^{n(x)} a(x),$$

with $n(x) = \nu_2(x + 1) \geq 1$ the Mersenne tail length and $a(x)$ the odd prefix factor. The exit exponent $r(x) = \nu_2(3^{n(x)}a(x) - 1)$ records how many powers of 2 are gained when one collapses a full block, and the block map satisfies the exact ratio identity

$$\frac{B(x)}{x} = \frac{3^{n(x)}}{2^{n(x)+r(x)}}.$$

Thus long tails $n(x)$ are exponentially expansionary (via $3^{n(x)}$), while large exit exponents $r(x)$ pull in the opposite direction through extra powers of 2 in the denominator.

From this perspective, two kinds of “pathological” behavior suggest themselves:

1. **Infinite tail mimicry.** A trajectory might try to emulate an infinite Mersenne tail pattern, in the sense that $n(x_k) \rightarrow \infty$ along the B -orbit $x_0, x_1 = B(x_0), x_2 = B^2(x_0), \dots$
2. **Symbolic shadowing.** The symbolic parameters $(n(x_k), a(x_k), r(x_k))$, viewed as a sequence in a countable alphabet, might become eventually periodic or track a nontrivial symbolic cycle.

Either phenomenon would amount to a kind of *shadowing*: the integer orbit would be tracking an “idealized” object (an infinite tail or a symbolic cycle) without ever converging to it. Motivated by both the residue-graph structure and the statistical behavior of Mersenne parameters from Part II, we formulate the following guiding principle.

No-shadowing principle (informal). *No nontrivial Collatz orbit under the block map B can indefinitely shadow an infinite Mersenne tail or an eventually periodic symbolic pattern. Along every orbit, the tail lengths $n(x_k)$ remain bounded and the symbolic parameter sequence $(n(x_k), a(x_k), r(x_k))$ is not eventually periodic, unless the orbit terminates at $x = 1$.*

A precise formulation of this principle, together with supporting structural results and a consolidated heuristic argument, is given in Appendix A. For the purposes of the main certificate-based development, the no-shadowing principle plays an *optional* role: it is not used in the proof of the finite-certificate criterion, but it serves as a structural sanity check that rules out certain classes of speculative counterexamples based on symbolic self-similarity or tail mimicry.

6.4 Symbolic dynamics and forbidden patterns

Each Mersenne block is fully characterized by three parameters: the tail length $n = n(x)$, the odd factor $a = a(x)$, and the exit exponent $r = r(x)$. We may therefore interpret the block dynamics $x \mapsto B(x)$ as generating a symbolic sequence

$$(n_0, a_0, r_0), (n_1, a_1, r_1), (n_2, a_2, r_2), \dots,$$

where (n_k, a_k, r_k) are the parameters associated to the k -th Mersenne block in the trajectory of some initial x_0 .

Let

$$\mathcal{A} := \{(n, a, r) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0} : a \text{ odd}\}.$$

Every B -orbit on odd integers induces a sequence in $\mathcal{A}^{\mathbb{N}}$, and the deterministic dependence of each block on its predecessor induces a (complicated) self-map on this symbolic space.

It is natural to ask whether such symbolic sequences can eventually repeat—i.e., whether some nontrivial B -orbit could give rise to an eventually periodic tail

$$(n_k, a_k, r_k)_{k \geq K} = (n_{k+T}, a_{k+T}, r_{k+T})_{k \geq K}$$

for some $K \geq 0$ and period $T \geq 1$. Such an eventual period would correspond to a symbolic cycle or fixed point under the induced parameter map and would realize symbolic self-shadowing at the level of block types.

The no-shadowing principle asserts that this cannot happen, except for the trivial fixed orbit at $x = 1$. Appendix A reformulates this as a precise conjecture and discusses how it interacts with the residue-graph structure and the statistical distribution of Mersenne parameters. As with the informal version above, this symbolic viewpoint is not needed for the certificate criterion itself, but helps position the framework relative to dynamical and symbolic approaches to the $3x + 1$ problem.

6.5 Stopping times for C , S , and B

Definition 6.4 (Stopping times). Let $x_0 \geq 1$ be odd. Define:

- The *Collatz stopping time* $\tau_C(x_0)$ as the least $i \geq 0$ with

$$C^i(x_0) = 1.$$

- The *Syracuse stopping time* $\tau_S(x_0)$ as the least $j \geq 0$ with

$$S^j(x_0) = 1.$$

- The *block stopping time* $\tau_B(x_0)$ as the least $k \geq 0$ with

$$B^k(x_0) = 1.$$

If no such i, j, k exist, the corresponding stopping time is taken to be $+\infty$.

Lemma 6.5 (Hierarchy of stopping times). *Let $x_0 \geq 1$ be odd. Then:*

1. For $x_0 > 1$,

$$\tau_C(x_0) > \tau_S(x_0).$$

2. Always,

$$\tau_S(x_0) \geq \tau_B(x_0),$$

with equality $\tau_S(x_0) = \tau_B(x_0)$ if and only if every Mersenne block along the B -orbit

$$x_0, B(x_0), B^2(x_0), \dots, B^{\tau_B(x_0)-1}(x_0)$$

has length $n(x_r) = 1$.

In particular, for $x_0 > 1$ we always have

$$\tau_C(x_0) > \tau_S(x_0) \geq \tau_B(x_0),$$

and typically (whenever at least one block has $n(x_r) > 1$),

$$\tau_C(x_0) > \tau_S(x_0) > \tau_B(x_0).$$

Proof. (1) Let x_ℓ denote the odd values along the Syracuse orbit:

$$x_0, x_1 = S(x_0), x_2 = S^2(x_0), \dots, x_{\tau_S(x_0)} = 1.$$

For each $\ell < \tau_S(x_0)$ we have

$$3x_\ell + 1 = 2^{a_\ell} x_{\ell+1}, \quad a_\ell := \nu_2(3x_\ell + 1) \geq 1.$$

In the Collatz trajectory, the transition from x_ℓ to $x_{\ell+1}$ consists of one step $x \mapsto 3x + 1$ followed by a_ℓ steps of division by 2, for a total of $1 + a_\ell$ Collatz steps. Summing,

$$\tau_C(x_0) = \sum_{\ell=0}^{\tau_S(x_0)-1} (1 + a_\ell).$$

Since each $a_\ell \geq 1$, we have $1 + a_\ell \geq 2$ and hence

$$\tau_C(x_0) \geq 2 \tau_S(x_0).$$

If $x_0 > 1$ then $\tau_S(x_0) \geq 1$, so $\tau_C(x_0) \geq 2 > \tau_S(x_0)$.

(2) Let $x_r := B^r(x_0)$ for $r \geq 0$, and write $n_r := n(x_r)$ for the length of the Mersenne tail (hence the size of the block) at x_r . By definition,

$$B(x_r) = S^{n_r}(x_r).$$

Thus the number of Syracuse steps from x_r to x_{r+1} is n_r . If $\tau_B(x_0) = k < \infty$, then the total number of Syracuse steps to reach 1 is

$$\tau_S(x_0) = \sum_{r=0}^{k-1} n_r.$$

Each $n_r \geq 1$, so $\tau_S(x_0) \geq k = \tau_B(x_0)$, with equality if and only if $n_r = 1$ for all $0 \leq r \leq k - 1$. \square

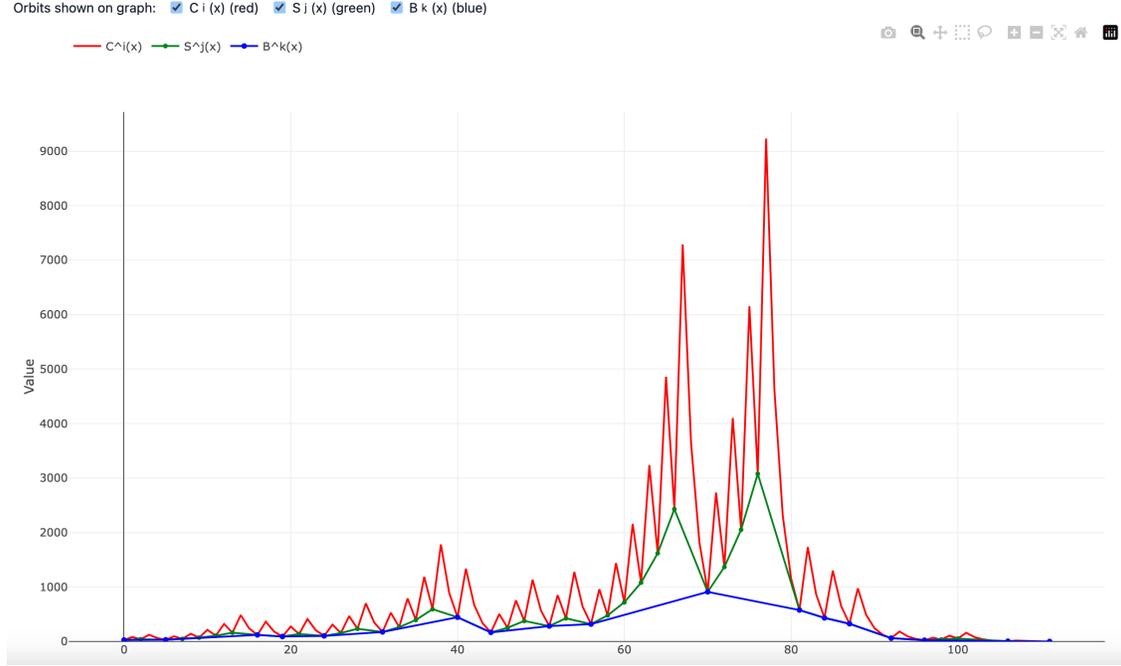


Figure 1: Sample orbits of C , S , and B for the classical starting value $x_0 = 27$, plotted against the underlying Collatz step index i . The Collatz orbit $C^i(27)$ (red) visits every iterate, the Syracuse orbit $S^j(27)$ (green) samples only the odd iterates, and the block orbit $B^k(27)$ (blue) records one representative point per Mersenne block.

Remark 6.6 (Time scales: C vs S vs B). Conceptually, the three maps operate at different time resolutions:

$$C \quad (\text{finest}), \quad S \quad (\text{odd-to-odd}), \quad B \quad (\text{block-to-block}).$$

Lemma 6.5 formalizes this: each level compresses several steps of the previous one, and the corresponding stopping times obey

$$\tau_C > \tau_S \geq \tau_B$$

for all nontrivial starting points, with strict inequalities in typical cases.

Remark 6.7 (Numerical illustration at $x_0 = 27$). Figure 1 and Table 1 concretely illustrate Lemma 6.5. ...

Remark 6.8 (Degenerate “no-wedge” class). The equality case $\tau_S(x_0) = \tau_B(x_0)$ in Lemma 6.5 is structurally degenerate and occurs if and only if every Mersenne block along the B -orbit of x_0 has length $n(x_r) = 1$, which is to say, if and only if the block map B and the Syracuse map S coincide along the entire orbit:

$$B^k(x_0) = S^k(x_0) \quad \text{for all } k \leq \tau_B(x_0).$$

Map f	$\tau_f(27)$	$\max_i f^i(27)$	$\tau_C(27)/\tau_f(27)$
C	111	9232	1
S	41	3077	≈ 2.71
B	17	911	≈ 6.53

Table 1: Stopping times and maximal values for the Collatz, Syracuse, and block orbits of $x_0 = 27$ corresponding to Fig. 1.

Equivalently, every odd iterate of the Syracuse orbit lies in the residue class 1 (mod 4), so the orbit never visits an odd integer $\equiv 3 \pmod{4}$. In this “trivial” class of cases, the Mersenne wedges degenerate, and the coarse-grained dynamics of B offers no time compression beyond the original Syracuse dynamics.

7 Summary and Transition to Statistical Structure

In this first part, we developed a new structural framework for analyzing Collatz dynamics through the lens of *Mersenne block decomposition*. By isolating the tail length $n(x) = \nu_2(x + 1)$ and odd prefix $a(x) = (x + 1)/2^{n(x)}$, we introduced the notion of a *Mersenne block*—a contiguous run of $n(x)$ strictly increasing odd iterates ending in a sharp descent step. This leads naturally to the definition of the block map $B(x) := S^{n(x)}(x)$, which skips entire blocks of intermediate Syracuse steps and captures the long-range structure of the orbit at a coarser resolution.

We identified key structural invariants: the tail length $n(x)$, the exit exponent $r(x)$, and the total stair height $J(x) = n(x) - 1$. The last value before descent, $L(x) = S^{n(x)-1}(x)$, and the block drop ratio $B(x)/x$ were shown to be governed by exact formulas involving these parameters. We also proved that block-to-block transitions are strictly downward in the intermediate value $L(x)$ for all $x \neq 1$, ruling out internal “shadowing” or infinite tail mimicry. In short, the original Collatz dynamics on \mathbb{Z}^+ has been reorganized into a finite-alphabet, directed system: each block is determined by its (n, r) parameters, and the global orbit is a walk through the space of these block types.

While this structural compression does not resolve the conjecture on its own, it transforms the problem from a chaotic, step-by-step iteration to a more structured evolution through block parameters. The question now is: how do these parameters behave across the integers?

This brings us to Part II, where we explore the *intrinsic statistics* of the tail lengths and exit exponents. We will show that $n(x)$ follows an exact geometric(1/2) distribution and that $r(x)$ appears to be statistically independent of $n(x)$ and also approximately geometric. This probabilistic viewpoint will lead to moment estimates for block ratios, expected logarithmic drift, and the identification of fractional Lyapunov functions that contract in expectation. These insights will motivate the analytic and certificate-based developments of Parts III–V.

Part II

Intrinsic Statistics

8 Intrinsic statistics of Mersenne block parameters

We now turn to the statistical behavior of the Mersenne block parameters

$$n(x) = \nu_2(x + 1), \quad r(x) = \nu_2(3^{n(x)}a(x) - 1),$$

and of the multiplicative factor $B(x)/x$.

8.1 Exact distribution of $n(x)$ on odd integers

Proposition 8.1 (Distribution of Mersenne-tail length $n(x)$). *Let x be uniformly chosen from odd integers in $\{1, 3, \dots, 2N - 1\}$. Then for each fixed $k \geq 1$,*

$$\Pr(n(x) = k) = 2^{-k}.$$

In particular, $n(x)$ has the geometric distribution with parameter $1/2$ on $\{1, 2, \dots\}$:

$$\Pr(n(x) = k) = 2^{-k}, \quad \mathbb{E}[n(x)] = \sum_{k \geq 1} k 2^{-k} = 2.$$

Proof. For an odd x , $x + 1$ is divisible by 2 but not by any higher power of 2 if and only if $x + 1 \equiv 2 \pmod{4}$, i.e. $x \equiv 1 \pmod{4}$. Among odd residues modulo 4, this is exactly half, so $\Pr(n(x) = 1) = 1/2$.

More generally, for $k \geq 1$, the event $n(x) \geq k$ is equivalent to $x + 1$ being divisible by 2^k , i.e. $x \equiv -1 \pmod{2^k}$. Among odd residues modulo 2^k , only one residue class (namely -1) satisfies this, so

$$\Pr(n(x) \geq k) = 2^{-(k-1)}.$$

Then

$$\Pr(n(x) = k) = \Pr(n(x) \geq k) - \Pr(n(x) \geq k + 1) = 2^{-(k-1)} - 2^{-k} = 2^{-k}.$$

□

Remark 8.2. Numerically, if one tabulates $n(x)$ for odd $x \leq 200,000$, the counts match this distribution exactly up to truncation: roughly half of all odds have $n(x) = 1$, a quarter have $n(x) = 2$, and so on.

8.2 Heuristic distribution of $r(x)$

In contrast to $n(x)$, the distribution of the exit exponent $r(x)$ is not determined purely by congruence conditions on x . However, the explicit formula

$$r(x) = \nu_2(3^{n(x)}a(x) - 1), \quad a(x) = \frac{x+1}{2^{n(x)}},$$

combined with simple 2-adic heuristics, strongly suggests that $r(x)$ has approximately the same geometric(1/2) distribution as $n(x)$.

Heuristic 8.3 (Distribution of $r(x)$). *Fix $n \geq 1$. For large x , the odd factor $a(x) = (x+1)/2^n$ behaves like a random odd residue modulo large powers of 2. Then $3^n a(x)$ is again a random odd residue modulo powers of 2 (since 3^n is invertible modulo 2^k), and $3^n a(x) - 1$ is a random even integer. Thus the valuation*

$$r(x) = \nu_2(3^n a(x) - 1)$$

should obey

$$\Pr(r(x) = k \mid n(x) = n) \approx 2^{-k}, \quad k \geq 1.$$

Consequently, the marginal distribution of $r(x)$ should be geometric(1/2):

$$\Pr(r(x) = k) \approx 2^{-k}, \quad \mathbb{E}[r(x)] \approx 2.$$

Remark 8.4. In numerical experiments (for odd $x \leq 200,000$), the empirical distribution of $r(x)$ is extremely close to geometric(1/2), with sample mean ≈ 2.00 and frequencies within rounding error of 2^{-k} . The empirical covariance of $(n(x), r(x))$ is very close to 0, suggesting that $n(x)$ and $r(x)$ are nearly independent in practice.

8.3 A random model for $(n(x), r(x))$

Motivated by Proposition 8.1 and Heuristic 8.3, we introduce the following random model for a “typical Mersenne block”.

Definition 8.5 (Independent geometric model for (n, r)). Let N and R be independent geometric random variables with parameter 1/2 on $\{1, 2, \dots\}$:

$$\Pr(N = n) = 2^{-n}, \quad \Pr(R = r) = 2^{-r}.$$

We interpret:

- N as the Mersenne-tail length $n(x)$ (equivalently, the wedge height or block length);
- R as the exit exponent $r(x)$.

Under this model, the approximate block-step factor is

$$\frac{B(x)}{x} \approx \frac{3^N}{2^{N+R}} = \left(\frac{3}{2}\right)^N 2^{-R}.$$

Remark 8.6. This model is not a proof about the exact behavior of B , but it is strongly supported by numerical data:

- the empirical distributions of $n(x)$ and $r(x)$ on odd integers up to 200,000 match the geometric(1/2) law very well;
- the empirical joint distribution of $(n(x), r(x))$ has almost zero covariance;
- the average logarithmic change in x per block step predicted by the model matches measured values to several decimal places (see below).

We therefore regard the independent geometric model as an accurate description of the *intrinsic statistics* of the Mersenne block map B .

8.4 Logarithmic drift of B

In the independent geometric model, the base-2 logarithmic increment per Mersenne block is

$$\Delta := \log_2 \frac{B(x)}{x} \approx N \log_2 \frac{3}{2} - R.$$

Proposition 8.7 (Expected logarithmic drift). *In the independent geometric model with $N, R \sim \text{Geom}(1/2)$, we have*

$$\mathbb{E}[\Delta] = 2 \log_2 \frac{3}{2} - 2 \approx -0.83007 < 0.$$

Proof. By independence,

$$\mathbb{E}[\Delta] = \mathbb{E}[N] \log_2 \frac{3}{2} - \mathbb{E}[R].$$

For a geometric(1/2) variable Y supported on $\{1, 2, \dots\}$ we have

$$\mathbb{E}[Y] = \sum_{k \geq 1} k 2^{-k} = 2.$$

Thus

$$\mathbb{E}[\Delta] = 2 \log_2 \frac{3}{2} - 2.$$

Since $\log_2(3/2) \approx 0.5849625$,

$$\mathbb{E}[\Delta] \approx 2 \cdot 0.5849625 - 2 \approx -0.830075.$$

□

Remark 8.8 (Arithmetic vs logarithmic contraction). The behavior of the Mersenne block map B in our intrinsic model mirrors the classical random-model heuristics for the Syracuse map S . In both settings one finds a striking “arithmetic vs logarithmic” dichotomy:

- In linear (arithmetic) scale, the one-step multiplicative factor is *critical*: in the independent geometric model for (n, r) one has $\mathbb{E}[B(x)/x] = 1$, and in the analogous random model for S one similarly has $\mathbb{E}[S(x)/x] \approx 1$.
- In logarithmic scale, the same models predict genuine contraction: for B we computed

$$\mathbb{E}[\log_2(B(x)/x)] = 2 \log_2 \frac{3}{2} - 2 \approx -0.83007 < 0,$$

and for S one obtains an analogous negative drift for $\log_2(S(x)/x)$.

This tension—critical in arithmetic mean but contractive in logarithmic mean—is the same phenomenon that underlies Tao’s use of logarithmic density in the analysis of typical Collatz orbits (see Tao’s work on almost-bounded Collatz orbits). In the Mersenne block framework it appears in a particularly clean form, with the block parameters $(n(x), r(x))$ having simple intrinsic statistics and the logarithmic drift per block given by an explicit closed formula.

Remark 8.9. Computing $\Delta(x) = \log_2 B(x) - \log_2 x$ for all odd $x \leq 200,000$ gives an empirical average of approximately -0.83013 , in excellent agreement with the theoretical value in Proposition 8.7. This strongly supports the independent geometric model for $(n(x), r(x))$.

Corollary 8.10 (Heuristic scaling of block stopping time). *Let $\tau_B(x)$ denote the block stopping time for x (the least $k \geq 0$ with $B^k(x) = 1$). In the independent geometric model, the typical number of block steps needed for $\log_2 x$ to decrease from L_0 to 0 satisfies*

$$\tau_B(x) \approx \frac{L_0}{-\mathbb{E}[\Delta]} \approx \frac{\log_2 x}{0.83007}.$$

Thus

$$\tau_B(x) \text{ grows on the order of } c_B \log_2 x, \quad c_B \approx 1.20,$$

at the Mersenne block level. Combined with $\mathbb{E}[n(x)] \approx 2$, this suggests that the Syracuse stopping time $\tau_S(x)$ grows on the order of a constant times $\log x$.

Remark 8.11. This agrees qualitatively with classical heuristics for the Collatz map (which also suggest logarithmic growth of stopping times), but here the analysis is carried out at the block level, where the intrinsic statistics are particularly clean.

8.5 Heuristic comparison of stopping times for S and B

Recall from Definition 6.4 that the Syracuse and block stopping times for an odd starting value x_0 are

$$\tau_S(x_0) := \min\{j \geq 0 : S^j(x_0) = 1\}, \quad \tau_B(x_0) := \min\{k \geq 0 : B^k(x_0) = 1\}.$$

Every step of B corresponds to a finite Mersenne block of S -steps. More precisely, if we write

$$x_r := B^r(x_0), \quad n_r := n(x_r),$$

then x_r is the start of block r , that block has length n_r , and

$$\tau_S(x_0) = \sum_{r=0}^{\tau_B(x_0)-1} n_r.$$

In the intrinsic statistics developed above, $n(x)$ is exactly geometric with parameter $1/2$ on the odd integers, with $\mathbb{E}[n(x)] = 2$, and numerical evidence suggests that along typical B -orbits the values $n_r = n(x_r)$ behave almost like independent samples from this distribution. Under this independence model, set

$$N_r \sim \text{Geom}(1/2) \text{ i.i.d.}, \quad \mathbb{E}[N_r] = 2,$$

and interpret N_r as the block length at step r . Conditional on $K := \tau_B(x_0)$, we then have

$$\tau_S(x_0) = \sum_{r=0}^{K-1} N_r, \quad \mathbb{E}[\tau_S(x_0) \mid K] = \sum_{r=0}^{K-1} \mathbb{E}[N_r] = 2K = 2\tau_B(x_0).$$

In other words, for large $\tau_B(x_0)$ we heuristically expect

$$\tau_S(x_0) \approx 2\tau_B(x_0).$$

Combining this with the logarithmic drift estimate for B from Proposition 8.7, which suggests

$$\tau_B(x_0) \approx c_B \log_2 x_0, \quad c_B \approx \frac{1}{0.83007} \approx 1.20,$$

we obtain the heuristic scaling laws

$$\tau_B(x_0) \approx 1.2 \log_2 x_0, \quad \tau_S(x_0) \approx 2.4 \log_2 x_0$$

for typical starting values x_0 . Thus the Mersenne block map B compresses the Syracuse time scale by a factor of about 2 on average: each block step hides roughly two odd-to-odd steps of the original accelerated Collatz map.

8.6 Upward vs downward block steps

The sign of Δ determines whether a block step moves up or down:

$$\Delta > 0 \iff N \log_2 \frac{3}{2} > R.$$

Proposition 8.12 (Frequency of upward and downward block steps). *In the independent geometric model,*

$$\Pr(\Delta > 0) \approx 0.28627, \quad \Pr(\Delta < 0) \approx 0.71373.$$

Proof sketch. We have

$$\Pr(\Delta > 0) = \sum_{n \geq 1} \Pr(N = n) \Pr\left(R < n \log_2 \frac{3}{2}\right).$$

With $N \sim \text{Geom}(1/2)$, $\Pr(N = n) = 2^{-n}$, and $R \sim \text{Geom}(1/2)$ gives $\Pr(R \leq m) = 1 - 2^{-m}$ for integer $m \geq 1$. Setting $m_n = \lfloor n \log_2(3/2) \rfloor$ and using $\Pr(R < m_n + 1) = 1 - 2^{-m_n}$, we obtain

$$\Pr(\Delta > 0) \approx \sum_{n \geq 1} 2^{-n} (1 - 2^{-m_n}).$$

Numerically evaluating this series gives $\Pr(\Delta > 0) \approx 0.28627$ and hence $\Pr(\Delta < 0) \approx 0.71373$. \square

Proof. Recall that in the independent geometric model we take N and R to be independent geometric(1/2) random variables on $\{1, 2, \dots\}$, so that

$$\Pr(N = n) = 2^{-n}, \quad \Pr(R = r) = 2^{-r}, \quad n, r \geq 1,$$

and we approximate the base-2 logarithmic increment per block by

$$\Delta \approx N \log_2 \frac{3}{2} - R.$$

In this model we treat Δ as the random variable

$$\Delta := N \log_2 \frac{3}{2} - R.$$

We first compute $\Pr(\Delta > 0)$. By independence,

$$\Pr(\Delta > 0) = \sum_{n \geq 1} \Pr(N = n) \Pr\left(R < n \log_2 \frac{3}{2}\right).$$

Since R takes integer values and $\log_2(3/2)$ is irrational, for each fixed n the event $\{R < n \log_2(3/2)\}$ is exactly

$$\{R \leq m_n\}, \quad m_n := \lfloor n \log_2(3/2) \rfloor \in \{0, 1, 2, \dots\}.$$

(If $m_n = 0$ this event is empty because $R \geq 1$.) Thus

$$\Pr\left(R < n \log_2 \frac{3}{2}\right) = \Pr(R \leq m_n).$$

For a geometric(1/2) variable R on $\{1, 2, \dots\}$ we have, for every integer $m \geq 0$,

$$\Pr(R \leq m) = \sum_{r=1}^m 2^{-r} = 1 - 2^{-m},$$

where the right-hand side is interpreted as 0 when $m = 0$. Therefore

$$\Pr\left(R < n \log_2 \frac{3}{2}\right) = 1 - 2^{-m_n},$$

and hence

$$\Pr(\Delta > 0) = \sum_{n \geq 1} 2^{-n} (1 - 2^{-m_n}), \quad m_n = \left\lfloor n \log_2 \frac{3}{2} \right\rfloor. \quad (3)$$

The series in (3) converges absolutely, since $|1 - 2^{-m_n}| \leq 1$ and $\sum_{n \geq 1} 2^{-n} = 1$. Truncating the sum after N terms gives

$$S_N := \sum_{n=1}^N 2^{-n} (1 - 2^{-m_n})$$

with tail

$$0 \leq \Pr(\Delta > 0) - S_N \leq \sum_{n > N} 2^{-n} = 2^{-N}.$$

For example, taking $N = 30$ yields

$$S_{30} \approx 0.2862745014, \quad 0 \leq \Pr(\Delta > 0) - S_{30} \leq 2^{-30} < 10^{-9},$$

so

$$\Pr(\Delta > 0) = 0.28627450\dots \approx 0.28627.$$

Next, we note that

$$\Pr(\Delta = 0) = 0.$$

Indeed, $\Delta = 0$ would require $N \log_2(3/2) = R$ for some integers $N, R \geq 1$, which would imply that $\log_2(3/2)$ is rational. But if $\log_2(3/2) = p/q$ with integers $p, q \geq 1$, then $2^p = 3^q$, which is impossible since the prime factorizations of 2^p and 3^q are distinct. Thus $\log_2(3/2)$ is irrational and the event $\{\Delta = 0\}$ has probability 0.

Consequently

$$\Pr(\Delta < 0) = 1 - \Pr(\Delta > 0) \approx 1 - 0.28627 = 0.71373.$$

This gives the claimed numerical values

$$\Pr(\Delta > 0) \approx 0.28627, \quad \Pr(\Delta < 0) \approx 0.71373.$$

□

Remark 8.13. Computing $\Delta(x)$ for odd $x \leq 200,000$ shows that approximately 28.6% of Mersenne block steps satisfy $B(x) > x$ and 71.4% satisfy $B(x) < x$, in excellent agreement with the model. Thus, in the intrinsic statistics of B , upward blocks are relatively rare but large when they occur, while downward blocks are more frequent.

9 Generating functions for block statistics

We now derive generating functions and moment generating functions for the statistics of N , R , and the approximate factor $B(x)/x$ in the independent geometric model.

9.1 Probability generating functions of N and R

Let N and R be independent geometric(1/2) variables on $\{1, 2, \dots\}$:

$$\Pr(N = n) = 2^{-n}, \quad \Pr(R = r) = 2^{-r}.$$

Lemma 9.1 (Generating functions of N and R). *The probability generating function of N is*

$$G_N(u) := \mathbb{E}[u^N] = \sum_{n \geq 1} 2^{-n} u^n = \frac{(u/2)}{1 - u/2},$$

valid for $|u| < 2$. Similarly

$$G_R(v) := \mathbb{E}[v^R] = \sum_{r \geq 1} 2^{-r} v^r = \frac{(v/2)}{1 - v/2},$$

for $|v| < 2$.

Proof. These are standard geometric series:

$$\sum_{n \geq 1} 2^{-n} u^n = \sum_{n \geq 1} \left(\frac{u}{2}\right)^n = \frac{u/2}{1 - u/2}.$$

The formula for G_R is identical. □

The joint generating function factorizes:

$$G_{N,R}(u, v) := \mathbb{E}[u^N v^R] = G_N(u) G_R(v) = \frac{(u/2)}{1 - u/2} \cdot \frac{(v/2)}{1 - v/2}.$$

9.2 Moment generating function of the multiplicative factor

Define the multiplicative factor

$$F := \frac{B(x)}{x} \approx \left(\frac{3}{2}\right)^N 2^{-R}.$$

We consider the (multiplicative) moment generating function

$$\varphi(\lambda) := \mathbb{E}[F^\lambda] = \mathbb{E}[(3/2)^{\lambda N} 2^{-\lambda R}].$$

Theorem 9.2 (Moment generating function of F). *In the independent geometric model,*

$$\varphi(\lambda) = \mathbb{E}[(B(x)/x)^\lambda] = \frac{\frac{(3/2)^\lambda}{2}}{1 - \frac{(3/2)^\lambda}{2}} \cdot \frac{2^{-1-\lambda}}{1 - 2^{-1-\lambda}},$$

for all λ with

$$\left| \frac{(3/2)^\lambda}{2} \right| < 1, \quad |2^{-1-\lambda}| < 1.$$

Proof. By independence,

$$\varphi(\lambda) = \mathbb{E}[(3/2)^{\lambda N}] \cdot \mathbb{E}[2^{-\lambda R}].$$

For the first factor, set $s_1 = (3/2)^\lambda$. Then

$$\mathbb{E}[(3/2)^{\lambda N}] = \sum_{n \geq 1} 2^{-n} s_1^n = \frac{s_1/2}{1 - s_1/2} = \frac{\frac{(3/2)^\lambda}{2}}{1 - \frac{(3/2)^\lambda}{2}},$$

provided $|s_1/2| < 1$, i.e. $|(3/2)^\lambda| < 2$.

For the second factor, set $s_2 = 2^{-\lambda}$. Then

$$\mathbb{E}[2^{-\lambda R}] = \sum_{r \geq 1} 2^{-r} s_2^r = \frac{s_2/2}{1 - s_2/2} = \frac{2^{-1-\lambda}}{1 - 2^{-1-\lambda}},$$

provided $|s_2/2| < 1$, i.e. $|2^{-1-\lambda}| < 1$.

Multiplying these expressions gives the claimed formula. □

Remark 9.3. The condition $|2^{-1-\lambda}| < 1$ holds for all real $\lambda > -1$ and is not restrictive for our purposes. The condition $|(3/2)^\lambda| < 2$ restricts λ to a finite interval around 0; for real λ this is equivalent to

$$-\frac{\log 2}{\log(3/2)} < \lambda < \frac{\log 2}{\log(3/2)},$$

which includes the interval $[0, 1]$ of primary interest.

9.3 Special values and Lyapunov exponents

From $\varphi(\lambda)$ we can read off several quantities of interest:

- $\varphi(0) = 1$ (trivial).
- $\varphi(1) = \mathbb{E}[F] = 1$:

$$\varphi(1) = \frac{\frac{3/2}{2}}{1 - \frac{3/2}{2}} \cdot \frac{2^{-2}}{1 - 2^{-2}} = \frac{3/4}{1 - 3/4} \cdot \frac{1/4}{1 - 1/4} = 3 \cdot \frac{1}{3} = 1.$$

- The derivative at $\lambda = 0$ satisfies

$$\varphi'(0) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathbb{E}[F^\lambda] = \mathbb{E}[\ln F],$$

so

$$\mathbb{E}[\log_2 F] = \frac{\mathbb{E}[\ln F]}{\ln 2} = \frac{\varphi'(0)}{\ln 2},$$

which reproduces the value in Proposition 8.7.

Proposition 9.4 (Negative drift for fractional powers). *For any real λ with $0 < \lambda < 1$, we have $\varphi(\lambda) < 1$ in the independent geometric model.*

Sketch. We noted that $\varphi(0) = 1$ and $\varphi(1) = 1$, and from Proposition 8.7 we know that

$$\varphi'(0) = \mathbb{E}[\ln F] < 0.$$

Thus $\varphi(\lambda) < 1$ for sufficiently small $\lambda > 0$. Numerically, one finds that $\varphi(\lambda) < 1$ throughout $(0, 1)$ and only returns to 1 at $\lambda = 1$. A rigorous proof can be obtained by analyzing the derivative $\varphi'(\lambda)$ from the explicit formula in Theorem 9.2, but we omit the algebra here. \square

Remark 9.5. Informally, this says that in the random model, the process $X_{k+1} = B(X_k)$ has the property that

$$\mathbb{E}[X_{k+1}^\lambda \mid X_k = x] \approx \varphi(\lambda) x^\lambda$$

with $\varphi(\lambda) < 1$ for $0 < \lambda < 1$. That is, x^λ behaves like a Lyapunov function *in expectation* for the Mersenne block dynamics.

Proof. Let $F > 0$ be the random multiplicative factor from the independent geometric model, so that

$$\varphi(\lambda) = \mathbb{E}[F^\lambda]$$

for all real λ in the domain of φ . From the previous bullet point in this subsection we already know that

$$\varphi(1) = \mathbb{E}[F] = 1.$$

Fix $0 < \lambda < 1$. Consider the function $f : (0, \infty) \rightarrow \mathbb{R}$ given by $f(t) = t^\lambda$. A direct computation shows

$$f''(t) = \lambda(\lambda - 1)t^{\lambda-2} < 0 \quad (t > 0),$$

so f is *strictly concave* on $(0, \infty)$.

Since $F \geq 0$ and $\mathbb{E}[F] = 1 < \infty$, we also have $\mathbb{E}[F^\lambda] < \infty$ for $0 < \lambda \leq 1$ (indeed $F^\lambda \leq 1 + F$ pointwise). Thus we may apply Jensen's inequality for the concave function f , obtaining

$$\varphi(\lambda) = \mathbb{E}[F^\lambda] = \mathbb{E}[f(F)] \leq f(\mathbb{E}[F]) = (\mathbb{E}[F])^\lambda = 1,$$

for every $0 < \lambda < 1$.

We now show that the inequality is in fact strict. In the independent geometric model we have

$$F = \left(\frac{3}{2}\right)^N 2^{-R}$$

with $N, R \sim \text{Geom}(1/2)$ independent on $\{1, 2, \dots\}$. In particular,

$$\Pr(N = 1, R = 1) > 0, \quad \Pr(N = 2, R = 1) > 0,$$

so

$$F = \left(\frac{3}{2}\right) \cdot 2^{-1} = \frac{3}{4} \quad \text{and} \quad F = \left(\frac{3}{2}\right)^2 \cdot 2^{-1} = \frac{9}{8}$$

each occur with positive probability. Thus F is not almost surely constant.

For a strictly concave function f , equality in Jensen's inequality holds only when the random variable is almost surely constant. Since F is not a.s. constant and $f(t) = t^\lambda$ is strictly concave for $0 < \lambda < 1$, the inequality above is strict:

$$\varphi(\lambda) = \mathbb{E}[F^\lambda] < (\mathbb{E}[F])^\lambda = 1$$

for every $0 < \lambda < 1$.

This proves $\varphi(\lambda) < 1$ for all real λ with $0 < \lambda < 1$, as claimed. \square

10 Conditioned statistics along block orbits

So far we have considered the distribution of $(n(x), r(x))$ when x is sampled uniformly from odd integers. We now discuss their distribution along actual orbits of the Mersenne block map.

10.1 Block orbits and empirical distributions

Fix an odd seed $x_0 \geq 1$ and consider its B -orbit:

$$x_{k+1} = B(x_k), \quad x_0 \text{ given.}$$

We record the corresponding Mersenne-tail lengths and exit exponents:

$$n_k := n(x_k) = \nu_2(x_k + 1), \quad r_k := r(x_k) = \nu_2(3^{n_k} a(x_k) - 1).$$

For each K , we can form empirical distributions

$$\hat{p}_K(n) := \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{1}_{\{n_k=n\}}, \quad \hat{q}_K(r) := \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{1}_{\{r_k=r\}}.$$

Heuristic 10.1 (Stationarity of block statistics). *For a typical seed x_0 , as $K \rightarrow \infty$ (assuming the orbit does not diverge), the empirical distributions \hat{p}_K and \hat{q}_K converge to the geometric(1/2) laws*

$$\hat{p}_K(n) \rightarrow 2^{-n}, \quad \hat{q}_K(r) \rightarrow 2^{-r},$$

and the empirical joint distribution of (n_k, r_k) converges to the product measure.

Remark 10.2. In numerical experiments with random seeds $x_0 \leq 10^6$ and K up to a few thousand, the Mersenne-tail lengths n_k along the B -orbits show a distribution very close to geometric(1/2), as do the r_k . Deviations tend to shrink as we increase sample sizes. This supports Heuristic 10.1, although a rigorous proof would require delicate analysis of the mixing properties of B modulo powers of 2.

10.2 Role of mixing modulo powers of 2

Formally, the distribution of $n(x)$ and $r(x)$ is governed by congruence properties of x modulo large powers of 2:

- $n(x) = k$ if and only if $x \equiv -1 \pmod{2^k}$ but $x \not\equiv -1 \pmod{2^{k+1}}$.
- $r(x) = \ell$ if and only if $3^{n(x)}a(x) - 1 \equiv 0 \pmod{2^\ell}$ but not modulo $2^{\ell+1}$.

If one could show that the map B acts in a suitably mixing way on odd integers modulo 2^m for each fixed m (for example, that its action is equidistributed among residue classes in some sense), then the geometric distributions for $n(x)$ and $r(x)$ would follow along typical orbits, not just from static sampling.

At present, such mixing properties are beyond reach, but the intrinsic statistics we see give a clear target for what one might hope to prove.

11 A Lyapunov-type viewpoint

We now interpret the intrinsic statistics of B in terms of Lyapunov functions and drift conditions.

11.1 Negative drift of $\log x$

11.2 The Collatz and Syracuse maps

From Proposition 8.7, in the independent geometric model we have

$$\mathbb{E}[\Delta] = \mathbb{E}\left[\log_2 \frac{B(x)}{x}\right] \approx -0.83007.$$

At an informal level, this means that, on average, each Mersenne block step reduces $\log_2 x$ by a constant amount. If the random model faithfully captures the true behavior of $(n(x), r(x))$, then along a B -orbit

$$\log_2 x_{k+1} \approx \log_2 x_k + \Delta_k, \quad \mathbb{E}[\Delta_k] \approx -0.83,$$

so loosely speaking

$$\log_2 x_k \approx \log_2 x_0 - 0.83 k,$$

up to fluctuations. This gives the heuristic scaling

$$\tau_B(x_0) \approx \frac{\log_2 x_0}{0.83}.$$

Remark 11.1. From the perspective of stochastic processes, this says the function $V(x) = \log_2 x$ has negative drift under the random model of B :

$$\mathbb{E}[V(B(x)) - V(x)] \approx -0.83.$$

The variance of the increment Δ is sizeable, so trajectories can exhibit large temporary growth, but the long-term tendency is downward.

11.3 Fractional power Lyapunov functions

Consider the function $V_\lambda(x) := x^\lambda$ for $0 < \lambda < 1$. In the independent geometric model,

$$\mathbb{E}[V_\lambda(B(x))/V_\lambda(x)] = \mathbb{E}[(B(x)/x)^\lambda] = \varphi(\lambda),$$

with $\varphi(\lambda)$ given by Theorem 9.2. As noted earlier, for $0 < \lambda < 1$ we have $\varphi(\lambda) < 1$.

Heuristic 11.2 (Supermartingale structure for fractional powers). *For any fixed $0 < \lambda < 1$, the process $X_{k+1} = B(X_k)$ approximately satisfies*

$$\mathbb{E}[X_{k+1}^\lambda \mid X_k = x] \approx \varphi(\lambda) x^\lambda,$$

with $\varphi(\lambda) < 1$. Thus $x \mapsto x^\lambda$ acts like a supermartingale for the Mersenne block dynamics: on average, each block step decreases x^λ by a constant factor.

Remark 11.3. In a fully rigorous setting, one would like to show that for some $\lambda > 0$,

$$\mathbb{E}[X_{k+1}^\lambda \mid X_k = x] \leq c(\lambda) X_k^\lambda$$

with $c(\lambda) < 1$ uniformly over all x in some domain. Our independent geometric model and the numerical evidence suggest this is true in practice, but proving it for the exact Mersenne block map B appears as hard as the underlying Collatz problem.

12 Implications for the Collatz problem

We briefly summarize what the structural and statistical analysis of Mersenne blocks tells us about the Collatz conjecture, and why it naturally leads to the finite-certificate framework developed in the next part.

12.1 Hierarchy of time scales revisited

As emphasized earlier, there are three natural maps:

- the original Collatz map C on \mathbb{N} ;
- the Syracuse map S on odd integers;
- the Mersenne block map B on odd integers.

For each odd starting value x_0 , the corresponding stopping times $\tau_C(x_0)$, $\tau_S(x_0)$, and $\tau_B(x_0)$ were defined in Definition 6.4. Lemma 6.5 shows that for $x_0 > 1$,

$$\tau_C(x_0) > \tau_S(x_0) \geq \tau_B(x_0).$$

Thus C is the finest time scale (Collatz steps), S is a compressed odd-to-odd view, and B is a further compression that jumps whole Mersenne blocks at a time.

The intrinsic statistics developed in this part of the manuscript add a probabilistic layer to this hierarchy: they suggest that, at the block level, the map B has a particularly simple statistical description with negative logarithmic drift.

12.2 Block formulation and probabilistic picture

In Mersenne block language, the Collatz conjecture is equivalent to the pure block statement

$$\text{for every odd } x_0 \geq 1 \text{ there exists } k \geq 0 \text{ with } B^k(x_0) = 1.$$

Structurally, each block at x is controlled by the pair $(n(x), r(x))$, where

$$n(x) = \nu_2(x + 1), \quad r(x) = \nu_2(3^{n(x)}a(x) - 1), \quad a(x) = \frac{x + 1}{2^{n(x)}}.$$

Part I on Structural Fundamentals also showed that $n(x)$ is the block length and $r(x)$ is the exit exponent, and that the multiplicative factor across a block is essentially

$$\frac{B(x)}{x} \approx \frac{3^{n(x)}}{2^{n(x)+r(x)}}.$$

Part II on Intrinsic Statistics has established:

- For x sampled uniformly from large ranges of odd integers, $n(x)$ is *exactly* geometric with parameter $1/2$.
- Empirically, $r(x)$ is also very close to geometric($1/2$) and nearly independent of $n(x)$.
- In the independent geometric model, one has

$$\mathbb{E}\left[\log_2 \frac{B(x)}{x}\right] = 2 \log_2 \frac{3}{2} - 2 \approx -0.83007 < 0,$$

so B is critical in linear scale (arithmetic mean of $B(x)/x$ is ≈ 1) but strongly contractive in logarithmic scale.

Under this model, the block stopping time scales like a constant times $\log x_0$, and the Syracuse stopping time scales like a constant times $\log x_0$ as well, with

$$\tau_B(x_0) \approx c_B \log_2 x_0, \quad \tau_S(x_0) \approx 2 c_B \log_2 x_0$$

for a constant $c_B \approx 1.2$. Fractional powers x^λ and $\log x$ behave like Lyapunov functions *in expectation* for the block dynamics. This paints a very coherent probabilistic picture: at the block level, the intrinsic statistics of $(n(x), r(x))$ strongly suggest that typical orbits of B (hence, of S and C) trend downwards in logarithmic scale and should eventually reach 1.

12.3 Stochastic vs. Block-Deterministic Modeling of Collatz Dynamics

A key distinction between Tao’s approach and the Mersenne Block Dynamics Framework (MBDF) lies in how the Collatz iteration is modeled: as a largely stochastic process versus a block-deterministic one. Tao’s 2022 work [13] effectively treats the accelerated Collatz (Syracuse) map S on odd integers as a type of random walk, using entropy and measure-theoretic methods to handle its unpredictable behavior. In particular, one imagines that at each odd iterate, the exponent $a = \nu_2(3x + 1)$ of 2 dividing $3x + 1$ (i.e., the number of halving steps following a $3x + 1$ jump) behaves like a random variable.

Empirically, a follows a geometric distribution with mean 2, so one “expects” about one division by 2 per odd step. However, the Syracuse map is *not* statistically neutral: for instance, $S(x)$ is twice as likely to be congruent to 2 (mod 3) as to 1 (mod 3) for a random odd x , and more generally, the distribution of $S^n(x) \bmod 3^n$ is highly non-uniform. As Tao explains, the uniform measure on a large interval is not transported by the Collatz dynamics to anything like a uniform measure on smaller intervals. This bias undermines naive attempts to iterate probabilistic descent arguments, forcing Tao to introduce a carefully chosen weight on the natural numbers (logarithmic density) to regain a form of invariance.

Using an entropy-based analysis on this weighted space, Tao proves that “almost all” orbits attain bounded values infinitely often. Equivalently, for any unbounded function $f(N)$, one has $\min_{n \geq 0} S^n(N) < f(N)$ for almost all N in the sense of logarithmic density. This is the strongest general Collatz result to date, but it still falls short of a full resolution: it does not guarantee convergence of every orbit, and the qualifier “almost all” refers to a weakened density notion. The core limitation is that treating Collatz steps as independent random events is only heuristically accurate. Subtle distributional irregularities accumulate over long orbits, making it difficult to propagate probabilistic control to global time scales.

MBDF approaches the problem differently by imposing deterministic structure on top of the Syracuse iteration, thereby addressing this statistical impasse. Rather than viewing each odd step as a memoryless random event, MBDF groups the dynamics into structured *blocks* governed by the binary expansion of the numbers. As shown in Part I (Section 2, ff.), each odd x has a well-defined “Mersenne tail” length $n(x) = \nu_2(x + 1)$ and exit exponent $r(x) = \nu_2(3^{n(x)}a(x) - 1)$, and the Collatz orbit can be coarse-grained into a sequence of blocks with parameters (n, r) . This yields a finite-alphabet deterministic dynamical system:

the global orbit becomes a walk through the space of block types (n, r) . All the randomness (and bias) of the $3x + 1$ process is now encapsulated in how these block parameters evolve, which MBDF constrains through number-theoretic invariants.

Notably, by classifying the admissible residue classes modulo 64 (see Proposition 15.2), we prove that the block drop exponent $s_{\text{exit}} = 1 + r(x)$ can only take values in the finite set $\{2, 3, 4, 5, 6\}$. This implies that the amount of dyadic descent per block is uniformly bounded above, a constraint incompatible with the heavy-tailed behavior one would expect from an unconstrained random walk model, and one that cannot be captured within a purely stochastic framework. This kind of deterministic constraint eliminates the long-range uncertainty present in Tao’s picture. It ensures, for instance, that every Collatz block, no matter how large x is, must relinquish at least a factor of 2^2 at its conclusion. Features like this bias, which are now made concrete and finite, are leveraged in the block framework to guarantee cumulative descent.

We explicitly compute a block-wise “ledger” of net logarithmic change ΔI in height across each block (see Section 14.1), and prove that, on the whole, this ledger trends negative. In fact, under the empirical independent-geometric law for (n, r) , one finds an expected logarithmic shrinkage of about -0.83 bits per block, indicating strong contraction on average. This phenomenon mirrors the negative drift for $\log_2 S(x)$ observed in the classical heuristics and in Tao’s work (the basis for his logarithmic density method). The crucial difference is that MBDF provides a rigorous mechanism to *enforce* this drift deterministically. By encoding the orbit as a sequence of (n, r) states and controlling state transitions via modular arithmetic (see Part IV, especially Section 15), MBDF sidesteps the issue of non-invariance that complicates Tao’s measure-theoretic approach.

In the framework developed in Part III (see Theorem 19.5), we show that one can trap the Collatz dynamics in a “descending” regime by verifying a finite set of conditions: specifically, a certain explicit inequality involving a residue-based invariant $A_{\min}^{(W)}$ must hold over any window of W consecutive blocks. If this finite criterion (together with a finite check of small cases) is satisfied, then *every* Collatz orbit must eventually descend to 1. In essence, the stochastic picture of Collatz is replaced by a finite-state, combinatorial one in which long-term behavior is completely constrained by a manageable residue graph and elementary inequalities.

This block-deterministic model not only reproduces the heuristic descent phenomenon underlying Tao’s result but elevates it from a statistical tendency to a concrete, verifiable condition. Thus, MBDF resolves the core limitation of the probabilistic approach, namely, the non-uniform “randomness” of Collatz steps, by absorbing it into a deterministic framework, ultimately providing a pathway to overcome the statistical impasse that had so far restricted progress to “almost all” orbits. In short, modeling Collatz via Mersenne blocks marries probabilistic insight with strict structural control, yielding a more powerful handle on the problem than the purely stochastic viewpoint.

12.4 From heuristic drift to a finite-certificate criterion

The statistical arguments above are intrinsically heuristic: they rely on random models and average behavior. They do not by themselves rule out the possibility of extremely rare, structured orbits that evade the negative drift suggested by the statistics.

Part III of this manuscript, beginning with Section 13, takes a completely deterministic turn. Therein we:

- construct an explicit height function $H(x)$ on odd integers;
- derive sharp, valuation-dependent bounds for the single-step change $H(S(x)) - H(x)$ along actual Syracuse steps;
- bundle these per-step bounds across whole Mersenne blocks to obtain a block-wise \log_6 -ledger depending only on the block parameters $(n(x), r(x))$;
- combine this ledger with residue information to formulate a *finite-certificate criterion*: a theorem saying that if a certain explicitly described finite object (a certificate) exists and satisfies finitely many checkable conditions, then the Collatz conjecture follows.

Thus the intrinsic statistics suggest *why* Collatz should be true at the block scale, while the finite-certificate framework in Part III shows *what* one would need to check, in a purely finite and deterministic way, to turn that intuition into a proof.

Part III

A Finite Certificate Criterion

13 Per-step analytic bounds for $H(x)$

In the intrinsic-statistics and Lyapunov-type discussions above, we analyzed the Mersenne block map B at a coarse scale: we modeled its block parameters $(n(x), r(x))$ by independent geometric laws and showed that, in that model, functions such as x^λ and $\log x$ exhibit strictly negative drift on average. This strongly suggests that the true Collatz dynamics should contract on the logarithmic scale, but those arguments, as noted above, live at the level of expectation and random models. From this point on, our approach is purely deterministic.

13.1 The Collatz and Syracuse maps

We construct an explicit height function

$$H(x) := \log_2 \left(x + \frac{1}{3} \right)$$

on odd integers and derive sharp inequalities for the single-step change $H(S(x)) - H(x)$ along actual Syracuse steps

$$x \mapsto x^+ := S(x) = \frac{3x+1}{2^{s(x)}}, \quad s(x) := \nu_2(3x+1).$$

The reason for working at the Syracuse scale rather than directly with B is that $B(x)$ bundles together a whole Mersenne block of odd-to-odd steps:

$$B(x) = S^{n(x)}(x),$$

and the fine control we need on H depends on the individual valuations $s(x)$ at each of those steps. At the level of a single Syracuse step we can write down an exact closed form for $H(S(x)) - H(x)$, split it into a valuation part and a small “tail” correction, and obtain uniform bounds depending only on $s(x)$. In later sections, these per-step bounds will be summed across the $n(x) - 1$ stair steps and the final exit step inside each Mersenne block, yielding the block-wise \log_6 -ledger $\Delta_I(J, s_{\text{exit}})$ used in the certificate criterion.

We continue to use $s(x) = \nu_2(3x+1)$ and the stair/exit terminology from Definition 5.1.

13.2 Height function and exact one-step identity

Definition 13.1 (Height function). For odd $x \geq 1$ define

$$H(x) := \log_2 \left(x + \frac{1}{3} \right).$$

Lemma 13.2 (Exact one-step identity). *Let x be odd with $s(x) = \nu_2(3x+1) \geq 1$ and $x^+ = S(x)$. Then*

$$H(x^+) - H(x) = \log_2 \left(\frac{3}{2^{s(x)}} \right) + \log_2 \left(1 + \frac{2^{s(x)}}{9(x + \frac{1}{3})} \right), \quad (4)$$

where $H(x) := \log_2(x + \frac{1}{3})$, as in Definition 13.1.

Proof. By definition,

$$x^+ = \frac{3x+1}{2^{s(x)}}.$$

Thus

$$x^+ + \frac{1}{3} = \frac{3x+1}{2^{s(x)}} + \frac{1}{3} = \frac{3(3x+1) + 2^{s(x)}}{3 \cdot 2^{s(x)}}.$$

Dividing by $x + \frac{1}{3}$ gives

$$\frac{x^+ + \frac{1}{3}}{x + \frac{1}{3}} = \frac{3(3x+1) + 2^{s(x)}}{3 \cdot 2^{s(x)}(x + \frac{1}{3})}.$$

But $3(3x + 1) = 9(x + \frac{1}{3})$, so

$$\frac{x^+ + \frac{1}{3}}{x + \frac{1}{3}} = \frac{9(x + \frac{1}{3}) + 2^{s(x)}}{3 \cdot 2^{s(x)}(x + \frac{1}{3})} = \frac{3}{2^{s(x)}} \left(1 + \frac{2^{s(x)}}{9(x + \frac{1}{3})} \right).$$

Now factor out $\frac{3}{2^{s(x)}}$:

$$\frac{9(x + \frac{1}{3}) + 2^{s(x)}}{3 \cdot 2^{s(x)}(x + \frac{1}{3})} = \frac{3}{2^{s(x)}} \cdot \frac{9(x + \frac{1}{3}) + 2^{s(x)}}{9(x + \frac{1}{3})} = \frac{3}{2^{s(x)}} \left(1 + \frac{2^{s(x)}}{9(x + \frac{1}{3})} \right).$$

Taking base-2 logarithms,

$$H(x^+) - H(x) = \log_2 \left(\frac{x^+ + \frac{1}{3}}{x + \frac{1}{3}} \right) = \log_2 \left(\frac{3}{2^{s(x)}} \right) + \log_2 \left(1 + \frac{2^{s(x)}}{9(x + \frac{1}{3})} \right),$$

which is the claimed identity. □

Lemma 13.3 (Collapsed form). *With notation as in Lemma 13.2,*

$$H(x^+) - H(x) = \log_2 \left(\frac{3}{2^{s(x)}} + \frac{1}{3x + 1} \right). \tag{5}$$

Proof. Since $9(x + \frac{1}{3}) = 3(3x + 1)$, the factor in parentheses in (4) simplifies to

$$\frac{3}{2^{s(x)}} \left(1 + \frac{2^{s(x)}}{9(x + \frac{1}{3})} \right) = \frac{3}{2^{s(x)}} + \frac{1}{3x + 1},$$

giving (5). □

Remark 13.4 (Motivation for the height function). At first sight the choice

$$H(x) := \log_2 \left(x + \frac{1}{3} \right)$$

may look ad hoc. It is in fact dictated by a simple algebraic observation about the affine map $x \mapsto 3x + 1$ and by the way we want to linearize Syracuse steps.

Indeed, the odd branch of the Collatz map is $x \mapsto 3x + 1$, and this affine map is conjugate to pure multiplication by 3 via the translation $x \mapsto x + \frac{1}{3}$:

$$3x + 1 = 3 \left(x + \frac{1}{3} \right).$$

When we pass to the odd-to-odd Syracuse map

$$S(x) = \frac{3x + 1}{2^{s(x)}}, \quad s(x) := \nu_2(3x + 1),$$

the same translation shows that

$$S(x) = \frac{3}{2^{s(x)}} \left(x + \frac{1}{3} \right),$$

so heuristically each step acts like multiplication by the factor $\frac{3}{2^{s(x)}}$ on the “shifted” coordinate $x + \frac{1}{3}$.

More generally, if we start from a shifted logarithmic height

$$H_c(x) := \log_2(x + c)$$

for an arbitrary constant c , a direct computation gives

$$H_c(x^+) - H_c(x) = \log_2 \left(\frac{3}{2^{s(x)}} \right) + \log_2 \left(1 + \frac{1 + c(2^{s(x)} - 3)}{3(x + c)} \right),$$

where $x^+ := S(x)$. The numerator $1 + c(2^{s(x)} - 3)$ is a pure multiple of $2^{s(x)}$ for all integers $s(x) \geq 1$ if and only if $c = \frac{1}{3}$, in which case it becomes $\frac{1}{3}2^{s(x)}$ and the second term simplifies to

$$\log_2 \left(1 + \frac{2^{s(x)}}{9(x + \frac{1}{3})} \right),$$

yielding the clean two-term identity in Lemma 13.2. Thus, up to an additive constant and the choice of logarithm base, $H(x) = \log_2(x + \frac{1}{3})$ is the unique logarithmic height of this form for which $H(S(x)) - H(x)$ splits into a main term depending only on $s(x)$ and a small tail of size $O(2^{s(x)}/x)$.

From this point of view H is not being pulled out of thin air; it is engineered so that the Syracuse step becomes “multiply by $\frac{3}{2^{s(x)}}$ plus a tiny correction” in the height coordinate. This is exactly what we need in order to aggregate the per-step drift into the block-wise \log_6 -ledger $\Delta_I(J, s_{\text{exit}})$ used in the certificate criterion.

Remark 13.5 (Valuation term vs. tail term). The collapsed expression

$$H(x^+) - H(x) = \log_2 \left(\frac{3}{2^{s(x)}} + \frac{1}{3x + 1} \right)$$

splits the step into:

- a valuation part $\frac{3}{2^{s(x)}}$, which decreases as $s(x)$ increases;
- a tail part $\frac{1}{3x+1}$, which decreases as x increases.

For fixed s , the worst (largest) increase in H occurs at the smallest x realizing that valuation.

13.3 Numerical constants and basic bounds

We record the constants we will use, together with their meaning:

$$\begin{aligned}
 A &:= \log_2 \frac{8}{5} \approx +0.67807, & \text{sharp max of } H(S(x)) - H(x) \text{ for } s(x) = 1 \text{ at } x = 3, \\
 C_2 &:= \log_2 \frac{11}{14} \approx -0.34792, & \text{sharp max for } s(x) = 2, x > 1, \text{ at } x = 9, \\
 B &:= \log_2 \frac{5}{8} \approx -0.67807, & \text{uniform bound for } s(x) \geq 3, \\
 B_4 &:= \log_2 \frac{7}{16} \approx -1.19265, & \text{uniform bound for } s(x) \geq 4, \\
 B_5 &:= \log_2 \frac{11}{32} \approx -1.54057, & \text{uniform bound for } s(x) \geq 5, \\
 B_6 &:= \log_2 \frac{19}{64} \approx -1.75207, & \text{uniform bound for } s(x) \geq 6.
 \end{aligned}$$

Lemma 13.6 (Sharp per-step maxima for $s = 1$ and $s = 2$). *Let x be odd.*

1. *If $s(x) = 1$ then $H(S(x)) - H(x) \leq A$, with equality at $x = 3$.*
2. *If $s(x) = 2$ and $x > 1$ then $H(S(x)) - H(x) \leq C_2$, with equality at $x = 9$. (For $x = 1$ we have $H(S(1)) - H(1) = 0$.)*

Proof. (1) If $s(x) = 1$ then $3x + 1 \equiv 2 \pmod{4}$, so $3x \equiv 1 \pmod{4}$ and hence $x \equiv 3 \pmod{4}$. Among odd x this means $x \equiv 3, 7 \pmod{8}$. The smallest such $x > 1$ is $x = 3$, for which $3x + 1 = 10$ and indeed $s(3) = 1$.

By Lemma 13.3,

$$H(S(x)) - H(x) = \log_2 \left(\frac{3}{2} + \frac{1}{3x+1} \right).$$

The right-hand side is maximized when $3x + 1$ is minimal among values with $s(x) = 1$, which occurs at $x = 3$, giving $3x + 1 = 10$. Thus

$$H(S(x)) - H(x) \leq \log_2 \left(\frac{3}{2} + \frac{1}{10} \right) = \log_2 \frac{8}{5} = A,$$

with equality at $x = 3$.

(2) If $s(x) = 2$ then $3x + 1$ is divisible by 4 but not by 8. The minimal odd x with this property is $x = 1$ (giving $3x + 1 = 4$), for which

$$H(S(1)) - H(1) = \log_2 \left(\frac{3}{4} + \frac{1}{4} \right) = 0.$$

The next candidate is $x = 9$, with $3 \cdot 9 + 1 = 28$ and $s(9) = 2$. Again using Lemma 13.3,

$$H(S(9)) - H(9) = \log_2 \left(\frac{3}{4} + \frac{1}{28} \right) = \log_2 \frac{11}{14} = C_2.$$

For $x > 9$ with $s(x) = 2$, the term $1/(3x + 1)$ is smaller, and the bound follows. \square

Lemma 13.7 (Uniform per-step bounds for large exits). *For odd x with $s(x) \geq 3$ we have*

$$H(S(x)) - H(x) \leq B = \log_2 \frac{5}{8}.$$

Moreover, if $s(x) \geq 4$ (respectively 5 or 6) we have progressively stronger bounds

$$H(S(x)) - H(x) \leq B_4, \quad B_5, \quad B_6,$$

respectively.

Proof. From Lemma 13.3,

$$H(S(x)) - H(x) = \log_2 \left(\frac{3}{2^{s(x)}} + \frac{1}{3x+1} \right).$$

If $s(x) \geq 3$ then $\frac{3}{2^{s(x)}} \leq \frac{3}{8}$ and, since $x \geq 1$, one has $\frac{1}{3x+1} \leq \frac{1}{4}$. Therefore

$$H(S(x)) - H(x) \leq \log_2 \left(\frac{3}{8} + \frac{1}{4} \right) = \log_2 \frac{5}{8} = B.$$

For $s(x) \geq 4, 5, 6$ we similarly obtain

$$\frac{3}{2^{s(x)}} + \frac{1}{3x+1} \leq \frac{3}{16} + \frac{1}{4}, \quad \frac{3}{32} + \frac{1}{4}, \quad \frac{3}{64} + \frac{1}{4},$$

giving the bounds B_4, B_5, B_6 . □

Theorem 13.8 (Per-step bounds, consolidated). *For odd $x \geq 1$ we have:*

$$\begin{aligned} \text{Stair: } s(x) = 1 &\Rightarrow H(S(x)) - H(x) \leq A, \\ \text{Exit } s = 2: s(x) = 2, x > 1 &\Rightarrow H(S(x)) - H(x) \leq C_2, \\ \text{Exit } s \geq 3: s(x) \geq 3 &\Rightarrow H(S(x)) - H(x) \leq B, \\ \text{Exit } s \geq 4, 5, 6: s(x) \geq m &\Rightarrow H(S(x)) - H(x) \leq B_m, \quad m = 4, 5, 6. \end{aligned}$$

Remark 13.9 (Ledger interpretation). We will use A as a cap on stair-induced growth per step, and the negative constants C_2, B, B_4, B_5, B_6 as lower bounds on exit-induced decrease per step. The stronger the exit valuation $s(x)$, the more negative the per-step change in H .

14 Block-wise \log_6 ledger

We now bundle the per-step bounds across a whole Mersenne block.

14.1 Block ledger entry

At the step level, each Syracuse move

$$x \mapsto x^+ = \frac{3x + 1}{2^{s(x)}}$$

contains exactly one factor of 3 and $s(x)$ factors of 2. Over a whole block at x with J stairs and an exit of valuation s , we therefore see

$$\text{total powers of 2 produced} = J + s, \quad \text{total powers of 3 used} = J + 1,$$

where the $J + 1$ counts the J stair steps plus the exit.

A natural way to keep track of this “2 vs. 3 struggle” is to look at the ratio

$$\frac{2^{J+s}}{3^{J+1}},$$

and to convert it to an additive quantity by taking a logarithm. We use base 6 because $6 = 2 \cdot 3$, so that powers of 2 and 3 combine cleanly on a single scale and align with the base-6 orbit prefix identities in Section 16.

Definition 14.1 (Block ledger entry). Let x be a block start with Mersenne-tail length $n(x)$ and exit exponent $r(x)$. Set

$$J(x) := n(x) - 1, \quad s_{\text{exit}}(x) := 1 + r(x).$$

We define the *block ledger entry* (or block gain in the \log_6 -ledger) by

$$\Delta_I(J(x), s_{\text{exit}}(x)) := \log_6 \frac{2^{J(x)+s_{\text{exit}}(x)}}{3^{J(x)+1}} = \log_6 \frac{2^{n(x)+r(x)}}{3^{n(x)}}. \quad (6)$$

More generally, for integers $J \geq 0$ and $s \geq 2$ we write

$$\Delta_I(J, s) := \log_6 \frac{2^{J+s}}{3^{J+1}}.$$

We say that a block with parameters (J, s) *wins* if $\Delta_I(J, s) > 0$, i.e. if

$$2^{J+s} > 3^{J+1}.$$

Remark 14.2 (Why base 6 and what it means to win). The condition $\Delta_I(J, s) > 0$ means that, in the 2-vs-3 ledger, the block produces more powers of 2 than are needed to compensate for its powers of 3: after J stairs and one exit, the downward effect of the 2^{J+s} divisors more than covers the upward effect of the 3^{J+1} multipliers. Measuring this on the \log_6 scale is convenient because each factor 2 contributes $+\log_6 2$ and each factor 3 contributes $+\log_6 3$, so

$$\Delta_I(J, s) = (J + s) \log_6 2 - (J + 1) \log_6 3.$$

When we sum over blocks, these contributions add linearly, and the same combination of 2's and 3's appears in the base-6 orbit prefix identity of Corollary 16.5. Equivalently, $\Delta_I(J, s) > 0$ is the logarithmic way of saying that the idealized core block factor $3^{J+1}/2^{J+s}$ is < 1 , so the block is contractive. A winning block is therefore one that, on this scale, contributes a net positive amount of “downward pressure” to the overall height function: its valuations are strong enough that, after the increase from the 3-steps, there is leftover contraction that can help compensate for increases from weaker blocks elsewhere in the orbit. There is no exactly “balanced” case with $\Delta_I(J, s) = 0$ since $2^{J+s} = 3^{J+1}$ has no integer solutions; every block either wins ($\Delta_I > 0$) or loses ($\Delta_I < 0$) in this ledger.

14.2 Single-block winning thresholds

Proposition 14.3 (Winning thresholds). *For each $J \geq 0$, define*

$$s_{\min}(J) := \min\{s \in \mathbb{Z}_{\geq 2} : \Delta_I(J, s) > 0\} = \min\{s \geq 2 : 2^{J+s} > 3^{J+1}\}.$$

Then

$$s_{\min}(J) = \lfloor \gamma + (\gamma - 1)J \rfloor + 1, \quad \gamma := \log_2 3.$$

In particular, for $J = 0, 1, 2, 3, 4, 5, 6, 7, 8$ one has:

J	0	1	2	3	4	5	6	7	8
$s_{\min}(J)$	2	3	3	4	4	5	6	6	7

Proof. The inequality $2^{J+s} > 3^{J+1}$ is equivalent to

$$J + s > (J + 1)\gamma \iff s > \gamma + (\gamma - 1)J,$$

where $\gamma = \log_2 3$. Thus the minimal integer s with $s \geq 2$ satisfying this is

$$s_{\min}(J) = \max\left\{2, \lfloor \gamma + (\gamma - 1)J \rfloor + 1\right\}.$$

One checks directly that for $J \geq 0$ the right-hand side is always ≥ 2 , giving the stated formula. The table follows by evaluation. □

Remark 14.4 (Quick consequences). Some simple corollaries:

- For $J = 0$ (i.e. $n(x) = 1$), every exit with $s \geq 2$ wins.
- For $J = 1$, $s \geq 3$ suffices (and $s = 2$ loses).
- For $J \geq 6$, any exit with $s \geq 6$ wins immediately.

These thresholds pair naturally with the residue forcing in Section 15, where $s \in \{2, 3, 4, 5, 6\}$.

15 Residue forcing and the mod-64 block graph

We now exploit low-modulus information to force exit valuations.

15.1 Exit valuation from the residue of the block peak

Theorem 15.1 (Exit valuation by $u \pmod{64}$). *Let u be odd and write $s(u) = \nu_2(3u + 1)$. Then the residue of u modulo 64 determines $s(u)$ as follows:*

$$s(u) = \begin{cases} 6, & u \equiv 21 \pmod{64}, \\ 5, & u \equiv 53 \pmod{64}, \\ 4, & u \equiv 5, 37 \pmod{64}, \\ 3, & u \equiv 13, 29, 45, 61 \pmod{64}, \\ 2, & u \equiv 1, 9, 17, 25, 33, 41, 49, 57 \pmod{64}, \\ 1, & \text{all remaining odd classes (never an exit)}. \end{cases}$$

Proof. Since we work modulo 64, it is enough to determine $s(u) = \nu_2(3u + 1)$ for the 32 odd residues $u \in \{1, 3, \dots, 63\}$. Any odd integer has a unique representative in this set, and for such u we have

$$4 \leq 3u + 1 \leq 3 \cdot 63 + 1 = 190 < 192 = 3 \cdot 64.$$

If $\nu_2(3u + 1) \geq 7$, then $3u + 1$ would be at least $2^7 = 128$ and a multiple of 128, hence equal to 128 or 256. The equation $3u + 1 = 128$ has no integer solution, and $3u + 1 = 256$ would force $u = 85 > 63$, which is not among our representatives. Thus for $1 \leq u \leq 63$ odd we have

$$1 \leq s(u) = \nu_2(3u + 1) \leq 6.$$

We now classify $s(u)$ by congruence classes of u modulo 64.

Step 1: reduction modulo 8. Write $u = 8q + r$ with $r \in \{1, 3, 5, 7\}$.

- If $r = 1$, then

$$3u + 1 = 3(8q + 1) + 1 = 24q + 4 = 4(6q + 1),$$

and $6q + 1$ is odd. Hence $s(u) = 2$ for all $u \equiv 1 \pmod{8}$. These are exactly the residue classes

$$u \equiv 1, 9, 17, 25, 33, 41, 49, 57 \pmod{64},$$

agreeing with the $s(u) = 2$ line of the theorem.

- If $r = 3$, then

$$3u + 1 = 3(8q + 3) + 1 = 24q + 10 = 2(12q + 5),$$

with $12q + 5$ odd. Thus $s(u) = 1$ for all $u \equiv 3 \pmod{8}$.

- If $r = 7$, then

$$3u + 1 = 3(8q + 7) + 1 = 24q + 22 = 2(12q + 11),$$

with $12q + 11$ odd. Thus $s(u) = 1$ for all $u \equiv 7 \pmod{8}$.

- If $r = 5$, then

$$3u + 1 = 3(8q + 5) + 1 = 24q + 16 = 8(3q + 2),$$

so

$$s(u) = 3 + \nu_2(3q + 2) \geq 3.$$

The eight residue classes with $u \equiv 5 \pmod{8}$ are

$$u \equiv 5, 13, 21, 29, 37, 45, 53, 61 \pmod{64}.$$

Thus:

$$s(u) = 2 \quad \text{iff} \quad u \equiv 1 \pmod{8},$$

$$s(u) = 1 \quad \text{iff} \quad u \equiv 3, 7 \pmod{8},$$

and the remaining classes $u \equiv 5 \pmod{8}$ are exactly those with $s(u) \geq 3$. It remains to determine $s(u)$ when $u \equiv 5 \pmod{8}$.

Step 2: the case $u \equiv 5 \pmod{8}$. Write $u = 8q + 5$ so that

$$3u + 1 = 8(3q + 2), \quad s(u) = 3 + \nu_2(3q + 2).$$

(a) q odd. Let $q = 2m + 1$. Then

$$3q + 2 = 3(2m + 1) + 2 = 6m + 5,$$

which is always odd. Thus $\nu_2(3q + 2) = 0$ and $s(u) = 3$. The condition $q \equiv 1, 3, 5, 7 \pmod{8}$ gives

$$u = 8q + 5 \equiv 13, 29, 45, 61 \pmod{64},$$

so

$$s(u) = 3 \quad \text{for} \quad u \equiv 13, 29, 45, 61 \pmod{64},$$

as stated.

(b) q even. Let $q = 2m$. Then

$$3q + 2 = 6m + 2 = 2(3m + 1),$$

so

$$s(u) = 3 + 1 + \nu_2(3m + 1) = 4 + \nu_2(3m + 1).$$

- If m is even, say $m = 2t$, then

$$3m + 1 = 6t + 1,$$

which is odd. Hence $\nu_2(3m + 1) = 0$ and $s(u) = 4$.

Here $q = 2m$ with m even means $q \equiv 0$ or $4 \pmod{8}$, so

$$u = 8q + 5 \equiv 5, 37 \pmod{64},$$

and we obtain

$$s(u) = 4 \quad \text{for} \quad u \equiv 5, 37 \pmod{64}.$$

- If m is odd, say $m = 2t + 1$, then

$$3m + 1 = 3(2t + 1) + 1 = 6t + 4 = 2(3t + 2),$$

so

$$\nu_2(3m + 1) = 1 + \nu_2(3t + 2), \quad s(u) = 4 + 1 + \nu_2(3t + 2) = 5 + \nu_2(3t + 2).$$

Now distinguish the parity of t .

- If t is even, $t = 2p$, then

$$3t + 2 = 6p + 2 = 2(3p + 1).$$

For p even, $3p + 1$ is odd, so $\nu_2(3t + 2) = 1$ and $s(u) = 6$. In our restricted range $1 \leq u \leq 63$, this occurs only for $u = 21$, and indeed

$$3 \cdot 21 + 1 = 64 = 2^6.$$

Thus $s(21) = 6$.

More generally, for $u \equiv 21 \pmod{64}$ one always has $\nu_2(3u + 1) \geq 6$, and the minimal value 6 is realized by $u = 21$ itself.

- If t is odd, $t = 2p + 1$, then

$$3t + 2 = 3(2p + 1) + 2 = 6p + 5,$$

which is odd, so $\nu_2(3t + 2) = 0$ and $s(u) = 5$.

In this case $q \equiv 6 \pmod{8}$, which yields

$$u = 8q + 5 \equiv 53 \pmod{64},$$

and for $u = 53$ we compute

$$3 \cdot 53 + 1 = 160 = 2^5 \cdot 5,$$

so $s(53) = 5$ as claimed. For all $u \equiv 53 \pmod{64}$ one has $\nu_2(3u + 1) \geq 5$, with the minimal value 5 attained at $u = 53$.

Collecting all cases, we obtain the table in the statement:

$$s(u) = \begin{cases} 6, & u \equiv 21 \pmod{64}, \\ 5, & u \equiv 53 \pmod{64}, \\ 4, & u \equiv 5, 37 \pmod{64}, \\ 3, & u \equiv 13, 29, 45, 61 \pmod{64}, \\ 2, & u \equiv 1, 9, 17, 25, 33, 41, 49, 57 \pmod{64}, \\ 1, & \text{all remaining odd classes.} \end{cases}$$

As noted at the beginning, for the canonical representatives $u \in \{1, 3, \dots, 63\}$ these are the exact values of $\nu_2(3u + 1)$; for other lifts in the same residue class, the valuation is never smaller and may be larger (e.g. for $u \equiv 21 \pmod{64}$), which only strengthens the exit step. This completes the proof. \square

Remark 15.2 (Meaning of “never an exit”). The classes with $s(u) = 1$ are precisely those where the next step is a stair. Exit nodes (block peaks) must fall in one of the $s \geq 2$ residue classes.

Remark 15.3 (Alternating Tail Blocks and Exit Step Bounds). Binary inputs ending in alternating bit patterns of the form $1010\dots 101$ (note that these are not Mersenne tails in the sense of a contiguous block of 1-bits) exhibit unusually large 2-adic valuations upon exit. Specifically, for such an alternating suffix of length $2k + 1$, the map $x \mapsto 3x + 1$ yields 2^{2k+2} , producing $\nu_2(3x + 1) = 2k + 2$. While such cases seem at first to contradict the established result that $s_{\text{exit}}(x) = 1 + \nu_2(3^{n(x)}a(x) - 1)$ is bounded within 2, 3, 4, 5, 6 under fixed residue conditions ($n(x) \pmod{8}, a(x) \pmod{32}$), they in fact do not.

These alternating-tail values fall into existing residue classes and do not expand the finite mod 64 residue graph. They represent “lifts” of states already encoded in the graph, experiencing larger-than-minimal exit exponents. For example, $x = 85 = 1010101_2$ satisfies $3x + 1 = 256 = 2^8$, but lies in the same residue state as $x = 21 = 10101_2$, which has $3x + 1 = 64 = 2^6$ and $s_{\text{exit}} = 6$. The larger exit exponent for $x = 85$ is not represented by a distinct edge; it merely overshoots the minimal forced drop and thus reinforces convergence.

Consequently, arbitrarily long alternating tails do not introduce new behaviors nor require adjustment to the finite-state model. The existing framework captures their structure, and their tendency to produce high exit valuations in a single step serves only to further accelerate descent.

15.2 Forced exit valuation by Mersenne parameters

Let x be a block start with $n(x) \geq 2$, and let $L(x)$ be its peak. From the structural formulas,

$$L(x) = 2 \cdot 3^{n(x)-1}a(x) - 1, \quad a(x) = \frac{x + 1}{2^{n(x)}}.$$

Thus

$$L(x) + 1 \equiv 2 \cdot 3^{n(x)-1}a(x) \pmod{64},$$

and the residue of $L(x)$ modulo 64 (hence $s(L(x))$) is controlled by $(n(x) \pmod{8}, a(x) \pmod{32})$.

Proposition 15.4 (Forced exit valuation from $(n \pmod{8}, a \pmod{32})$). *Let x be a block start with $n := n(x) \geq 2$ and $a := a(x)$ odd. Set*

$$r_n \equiv 3^{-(n-1)} \pmod{32},$$

where $3^{-(n-1)}$ denotes the inverse of 3^{n-1} modulo 32. Then the exit valuation $s_{\text{exit}}(x) = s(L(x)) \in \{2, 3, 4, 5, 6\}$ is forced by $a \pmod{32}$ via

$$s_{\text{exit}} = \begin{cases} 6 & \iff a \equiv r_n, \\ 5 & \iff a \equiv r_n + 16, \\ 4 & \iff a \equiv r_n \pm 8, \\ 3 & \iff a \equiv r_n \pm 4, r_n \pm 12, \\ 2 & \iff a \equiv r_n \pm 2, r_n \pm 6, r_n \pm 10, r_n \pm 14, \end{cases} \pmod{32}.$$

In each $n \pmod{8}$ slice there are exactly 1, 1, 2, 4, 8 classes forcing $s_{\text{exit}} = 6, 5, 4, 3, 2$.

%endproof

Proof. Let x be a block start with $n := n(x) \geq 2$ and $a := a(x)$ odd, and write $u := L(x)$. By Lemma 4.2 we have the structural formula

$$u = L(x) = 2 \cdot 3^{n-1}a - 1.$$

All congruences below are taken modulo 32 or 64 as indicated.

Step 1: Relating $u \pmod{64}$ to $3^n a \pmod{32}$. From the structural formula we obtain

$$u + 1 \equiv 2 \cdot 3^{n-1}a \pmod{64}.$$

Multiplying by 3 and then dividing by 2 gives

$$3(u + 1) \equiv 2 \cdot 3^n a \pmod{64} \implies 3^n a \equiv \frac{3(u + 1)}{2} \pmod{32}.$$

In particular, $3^n a \pmod{32}$ depends only on the residue class of $u \pmod{64}$. Since $\gcd(3^n, 32) = 1$, 3^n is invertible modulo 32; set

$$r_n \equiv 3^{-n} \pmod{32},$$

so that

$$a \equiv r_n \cdot \frac{3(u + 1)}{2} \pmod{32}.$$

Step 2: The classes of $t(u) := \frac{3(u + 1)}{2} \pmod{32}$. By Theorem 15.1, for an odd u we have $s(u) = \nu_2(3u + 1)$ determined by $u \pmod{64}$ as

$$s(u) = \begin{cases} 6, & u \equiv 21 \pmod{64}, \\ 5, & u \equiv 53 \pmod{64}, \\ 4, & u \equiv 5, 37 \pmod{64}, \\ 3, & u \equiv 13, 29, 45, 61 \pmod{64}, \\ 2, & u \equiv 1, 9, 17, 25, 33, 41, 49, 57 \pmod{64}, \end{cases}$$

and $s(u) = 1$ for all remaining odd classes.

Define

$$t(u) \equiv \frac{3(u+1)}{2} \pmod{32}.$$

A direct calculation for each residue class in the theorem gives:

- If $u \equiv 21 \pmod{64}$ (so $s(u) = 6$), then

$$u + 1 \equiv 22 = 2 \cdot 11, \quad t(u) \equiv 3 \cdot 11 = 33 \equiv 1 \pmod{32}.$$

Thus all $s = 6$ classes have $t(u) \equiv 1$.

- If $u \equiv 53 \pmod{64}$ (so $s(u) = 5$), then

$$u + 1 \equiv 54 = 2 \cdot 27, \quad t(u) \equiv 3 \cdot 27 = 81 \equiv 17 \pmod{32},$$

so $t(u) \equiv 17$.

- If $u \equiv 5, 37 \pmod{64}$ (so $s(u) = 4$), then

$$\begin{aligned} u \equiv 5 &\Rightarrow u + 1 \equiv 6 = 2 \cdot 3 \Rightarrow t(u) \equiv 3 \cdot 3 = 9, \\ u \equiv 37 &\Rightarrow u + 1 \equiv 38 = 2 \cdot 19 \Rightarrow t(u) \equiv 3 \cdot 19 = 57 \equiv 25. \end{aligned}$$

Thus for $s(u) = 4$ we have $t(u) \in \{9, 25\}$.

- If $u \equiv 13, 29, 45, 61 \pmod{64}$ (so $s(u) = 3$), then

$$\begin{aligned} u \equiv 13 &\Rightarrow u + 1 \equiv 14 = 2 \cdot 7 \Rightarrow t(u) \equiv 3 \cdot 7 = 21, \\ u \equiv 29 &\Rightarrow u + 1 \equiv 30 = 2 \cdot 15 \Rightarrow t(u) \equiv 3 \cdot 15 = 45 \equiv 13, \\ u \equiv 45 &\Rightarrow u + 1 \equiv 46 = 2 \cdot 23 \Rightarrow t(u) \equiv 3 \cdot 23 = 69 \equiv 5, \\ u \equiv 61 &\Rightarrow u + 1 \equiv 62 = 2 \cdot 31 \Rightarrow t(u) \equiv 3 \cdot 31 = 93 \equiv 29. \end{aligned}$$

Thus for $s(u) = 3$,

$$t(u) \in \{5, 13, 21, 29\}.$$

- If $u \equiv 1, 9, 17, 25, 33, 41, 49, 57 \pmod{64}$ (so $s(u) = 2$), then

$$\begin{aligned} u \equiv 1 &\Rightarrow u + 1 \equiv 2 = 2 \cdot 1 \Rightarrow t(u) \equiv 3 \cdot 1 = 3, \\ u \equiv 9 &\Rightarrow u + 1 \equiv 10 = 2 \cdot 5 \Rightarrow t(u) \equiv 3 \cdot 5 = 15, \\ u \equiv 17 &\Rightarrow u + 1 \equiv 18 = 2 \cdot 9 \Rightarrow t(u) \equiv 3 \cdot 9 = 27, \\ u \equiv 25 &\Rightarrow u + 1 \equiv 26 = 2 \cdot 13 \Rightarrow t(u) \equiv 3 \cdot 13 = 39 \equiv 7, \\ u \equiv 33 &\Rightarrow u + 1 \equiv 34 = 2 \cdot 17 \Rightarrow t(u) \equiv 3 \cdot 17 = 51 \equiv 19, \\ u \equiv 41 &\Rightarrow u + 1 \equiv 42 = 2 \cdot 21 \Rightarrow t(u) \equiv 3 \cdot 21 = 63 \equiv 31, \\ u \equiv 49 &\Rightarrow u + 1 \equiv 50 = 2 \cdot 25 \Rightarrow t(u) \equiv 3 \cdot 25 = 75 \equiv 11, \\ u \equiv 57 &\Rightarrow u + 1 \equiv 58 = 2 \cdot 29 \Rightarrow t(u) \equiv 3 \cdot 29 = 87 \equiv 23. \end{aligned}$$

Hence for $s(u) = 2$,

$$t(u) \in \{3, 7, 11, 15, 19, 23, 27, 31\}.$$

Grouping these, we have the following partition of the odd residues modulo 32:

$$t(u) \equiv \begin{cases} 1, & s(u) = 6, \\ 17, & s(u) = 5, \\ 9, 25, & s(u) = 4, \\ 5, 13, 21, 29, & s(u) = 3, \\ 3, 7, 11, 15, 19, 23, 27, 31, & s(u) = 2. \end{cases}$$

Step 3: From $t(u)$ to $a \pmod{32}$. Recall that

$$3^n a \equiv t(u) \pmod{32}, \quad a \equiv r_n t(u) \pmod{32},$$

with $r_n \equiv 3^{-n} \pmod{32}$. Since r_n is odd, multiplication by r_n is a bijection of the odd residue classes modulo 32. Therefore the sets of possible residues $a \pmod{32}$ for each valuation $s(u)$ are just the images of the above $t(u)$ -sets under multiplication by r_n .

We now check how these images look in *additive* form. Let r be an arbitrary odd residue modulo 32. Then:

- For $t = 1$ we have $rt \equiv r$.
- For $t = 17$ we use $17 = 1 + 16$ and $16r \equiv 16$ (since $16r = 16(2k + 1) \equiv 16 \pmod{32}$), hence

$$17r \equiv r + 16 \pmod{32}.$$

- For $t \in \{9, 25\}$ we write $9 = 1 + 8$, $25 = 1 - 8$. Then

$$9r = r + 8r, \quad 25r = r - 8r.$$

As r is odd, $8r = 8(2k + 1) = 16k + 8 \equiv 8$ or $24 \equiv -8$ modulo 32, so $\{+8r, -8r\} = \{+8, -8\}$. Hence

$$\{9r, 25r\} = \{r + 8, r - 8\} = \{r \pm 8\}.$$

- For $t \in \{5, 13, 21, 29\}$ we write

$$5 = 1 + 4, \quad 13 = 1 + 12, \quad 21 = 1 - 12, \quad 29 = 1 - 4.$$

Thus

$$5r = r + 4r, \quad 13r = r + 12r, \quad 21r = r - 12r, \quad 29r = r - 4r.$$

Now $4r = 4(2k + 1) = 8k + 4$ is congruent to 4, 12, 20, 28, i.e. 4, -4, 12, -12 modulo 32, and $12r = 3 \cdot 4r$ permutes $\{\pm 4, \pm 12\}$ as well. Therefore

$$\{\pm 4r, \pm 12r\} = \{\pm 4, \pm 12\},$$

and hence

$$\{5r, 13r, 21r, 29r\} = \{r \pm 4r, r \pm 12r\} = \{r \pm 4, r \pm 12\}.$$

- For $t \in \{3, 7, 11, 15, 19, 23, 27, 31\}$ we note that

$$t = 1 + d, \quad d \in \{\pm 2, \pm 6, \pm 10, \pm 14\}.$$

Let

$$D := \{\pm 2, \pm 6, \pm 10, \pm 14\} \subset \mathbb{Z}/32\mathbb{Z}.$$

Then $t \in 1 + D$ and $tr = r + dr$. Each $d \in D$ can be written as $d = 2k$ with k odd. Since multiplication by r permutes the odd residue classes modulo 16, the set

$$\{kr \bmod 16 : k \in \{\pm 1, \pm 3, \pm 5, \pm 7\}\}$$

is again the set of all odd residues modulo 16. Multiplying back by 2, we see that

$$\{dr \bmod 32 : d \in D\} = D.$$

Therefore

$$\{tr : t \in 1 + D\} = \{r + dr : d \in D\} = \{r \pm 2, r \pm 6, r \pm 10, r \pm 14\}.$$

Summarizing, for arbitrary odd r we have

$$r \cdot \{1\} = \{r\}, \quad r \cdot \{17\} = \{r + 16\}, \quad r \cdot \{9, 25\} = \{r \pm 8\},$$

$$r \cdot \{5, 13, 21, 29\} = \{r \pm 4, r \pm 12\}, \quad r \cdot \{3, 7, 11, 15, 19, 23, 27, 31\} = \{r \pm 2, r \pm 6, r \pm 10, r \pm 14\}.$$

Applying this with $r = r_n$ and with the $t(u)$ -sets from Step 2, we obtain the asserted classification of $a \bmod 32$ by exit valuation:

$$s_{\text{exit}}(x) = s(L(x)) = \begin{cases} 6 & \iff a \equiv r_n, \\ 5 & \iff a \equiv r_n + 16, \\ 4 & \iff a \equiv r_n \pm 8, \\ 3 & \iff a \equiv r_n \pm 4, r_n \pm 12, \\ 2 & \iff a \equiv r_n \pm 2, r_n \pm 6, r_n \pm 10, r_n \pm 14, \end{cases} \quad (\bmod 32).$$

Finally, in each fixed $n \bmod 8$ slice there are 16 odd residue classes $a \bmod 32$, and the above partition splits them into 1, 1, 2, 4, 8 classes forcing $s_{\text{exit}} = 6, 5, 4, 3, 2$ respectively. This completes the proof. \square

Remark 15.5 (Semantics of forced valuations). For fixed $(n \bmod 8, a \bmod 32)$, Proposition 15.4 determines the *minimal* exit valuation s_{exit} realized by some lift. Larger valuations s may occur for some lifts, but only strengthen the ledger. When building the residue graph we include only edges where this minimal exit valuation is actually realized by at least one lift.

15.3 The enriched mod-64 Mersenne-block graph

We now define the finite state space and edges.

Definition 15.6 (State types). We consider nodes encoding the Mersenne parameters modulo $(8, 32)$:

- $T_1(a_{32})$: tail length $n(x) = 1$, $a(x) \equiv a_{32} \pmod{32}$ (a_{32} odd). These are zero-stair blocks.
- $D(a_{32})$: tail length $n(x) = 2$, $a(x) \equiv a_{32} \pmod{32}$. These are one-stair (degenerate) blocks.
- $ND(n_8, a_{32})$: tail length $n(x) \geq 3$, recorded modulo 8 by $n_8 \in \{3, \dots, 10\}$ and $a(x) \pmod{32}$.

Definition 15.7 (Edges and realizability). From a state (n, a) with $n \geq 2$, we:

1. Use Proposition 15.4 to get a forced exit valuation $s_{\text{exit}} \in \{2, \dots, 6\}$.
2. Set $r = s_{\text{exit}} - 1$ and $M = 3^n a - 1$.
3. Enumerate the finite set of lifts $a_k = a_{32} + 32k$ needed to resolve $M \pmod{2^{r+2}}$.
4. For each lift:
 - (a) Check $\nu_2(3^n a_k - 1) = r$; lifts with larger valuation are excluded for the forced- s edge.
 - (b) Compute $B(x)$ via $B = (3^n a_k - 1)/2^r$ and then
$$n' := \nu_2(B + 1), \quad a' := \frac{B + 1}{2^{n'}}.$$
 - (c) Assign the target node by $(n' \pmod{8}, a' \pmod{32})$.
5. Deduplicate successors ($\text{src} \rightarrow \text{dst}$) with their forced s_{exit} and the associated stair count $J = n - 1$.

The resulting directed graph has nodes labeled by $(n \pmod{8}, a \pmod{32})$ and edges labeled by (J, s_{exit}) .

Remark 15.8 (Option A vs. Option B). We can either:

- (*Option A*) keep the $n = 1$ band (T_1) as explicit nodes. Each edge then corresponds to a single Mersenne block (one application of B).
- (*Option B*) compress across $n = 1$ nodes by collapsing their self-loops. Edges may then represent several consecutive blocks; the dynamic program must *unfold* them.

Option A is conceptually cleaner; Option B yields a smaller graph and slightly stronger invariants. The certificate formulation below works in either model.

16 Orbit-prefix identities and carry

We momentarily step away from block structure and consider arbitrary finite Syracuse orbit prefixes on odd integers.

Remark 16.1 (Two notions of prefix). Earlier in this manuscript we used the word “prefix” for the even bitwise prefix segment $P(x)$ and the odd bitwise prefix segment $a(x) = P(x) + 1$ in the Mersenne-tail decomposition

$$x = P(x)2^{n(x)} + (2^{n(x)} - 1),$$

where $(P(x))_2$ is the leading segment of binary digits followed by the Mersenne tail for $(x)_2$. Here we use “prefix” to refer to a finite initial segment of a Syracuse orbit,

$$x_0, x_1 = S(x_0), \dots, x_m = S^m(x_0),$$

which we distinguish as an *orbit prefix*. Given this clarification, context should suffice to discern whether we are referring, e.g., to the even prefix segment $P(x)$ for the bit-string for $(x)_2$, or to orbit-prefix identities and orbit-prefix carry for the trajectory-level constructions below.

16.1 Prefix telescoping

Let x_0 be odd and consider the odd-only Syracuse trajectory

$$x_0 \xrightarrow{s_0} x_1 \xrightarrow{s_1} x_2 \xrightarrow{s_2} \dots \xrightarrow{s_{m-1}} x_m,$$

where $x_{k+1} = S(x_k) = (3x_k + 1)/2^{s_k}$ and $s_k = \nu_2(3x_k + 1)$.

Definition 16.2 (Cumulative valuations). Define

$$G_m := \sum_{k=0}^{m-1} s_k, \quad g_m := \sum_{k=0}^{m-1} (s_k + 1) = G_m + m.$$

Proposition 16.3 (Prefix telescoping identity). *For every $m \geq 1$,*

$$2^{G_m} x_m = 3^m x_0 + \sum_{j=0}^{m-1} 3^{m-1-j} 2^{G_j}. \tag{7}$$

Proof. We use induction on m . For $m = 1$, $G_1 = s_0$ and

$$2^{G_1} x_1 = 2^{s_0} \frac{3x_0 + 1}{2^{s_0}} = 3x_0 + 1 = 3x_0 + 2^{G_0}.$$

Assume (7) holds for some $m \geq 1$, i.e.

$$2^{G_m} x_m = 3^m x_0 + \sum_{j=0}^{m-1} 3^{m-1-j} 2^{G_j}.$$

Then $G_{m+1} = G_m + s_m$ and

$$\begin{aligned}
2^{G_{m+1}} x_{m+1} &= 2^{G_m + s_m} \frac{3x_m + 1}{2^{s_m}} \\
&= 3 \cdot 2^{G_m} x_m + 2^{G_m} \\
&= 3 \left(3^m x_0 + \sum_{j=0}^{m-1} 3^{m-1-j} 2^{G_j} \right) + 2^{G_m} \\
&= 3^{m+1} x_0 + \sum_{j=0}^{m-1} 3^{m-j} 2^{G_j} + 2^{G_m} \\
&= 3^{m+1} x_0 + \sum_{j=0}^m 3^{m-j} 2^{G_j},
\end{aligned}$$

which is (7) with m replaced by $m + 1$. □

16.2 Base-3 and base-6 orbit-prefix identities

Definition 16.4 (Prefix carry). Define

$$R_m := \sum_{j=0}^{m-1} 3^{m-1-j} 2^{G_j}, \quad \rho_m := \frac{R_m}{3^m x_0}.$$

Corollary 16.5 (Base-3 and base-6 orbit-prefix identities). *For every $m \geq 1$,*

$$\log_3 \left(\frac{2^{G_m}}{x_0} \right) = m - \log_3 x_m + \log_3(1 + \rho_m),$$

and

$$\log_6 \left(\frac{2^{g_m}}{x_0} \right) = m - \log_6 x_m + \log_6(1 + \rho_m).$$

Equivalently,

$$\frac{2^{g_m}}{x_0} = \frac{6^m}{x_m} (1 + \rho_m). \tag{8}$$

Proof. Divide (7) by x_0 and x_m , and then take base-3 and base-6 logarithms. Passing from base 3 to base 6 uses $g_m = G_m + m$ and multiplying numerator and denominator by 2^m . □

Proposition 16.6 (Carry recursion). *For all $m \geq 0$,*

$$\rho_{m+1} - \rho_m = \frac{1 + \rho_m}{3x_m}.$$

Proof. From $R_{m+1} = 3R_m + 2^{G_m}$ we obtain

$$\rho_{m+1} = \frac{R_{m+1}}{3^{m+1}x_0} = \frac{3R_m + 2^{G_m}}{3^{m+1}x_0} = \rho_m + \frac{2^{G_m}}{3^{m+1}x_0}.$$

Using $2^{G_m}x_m = 3^m x_0(1 + \rho_m)$ (from the base-3 identity) gives

$$\frac{2^{G_m}}{3^{m+1}x_0} = \frac{1 + \rho_m}{3x_m},$$

as claimed. □

Remark 16.7 (What the carry captures). The orbit-prefix identities show that the cumulative effect of all rounding/correction across m steps is captured by a single scalar ρ_m . Over a structured window of steps (for example, W consecutive Mersenne blocks), only the ratio $(1 + \rho_{m+S})/(1 + \rho_m)$ over that window will matter.

17 Window identity and carry-controlled criterion

We now combine the block ledger with the prefix identities to obtain a window inequality.

17.1 Exact window identity

Consider a window of W consecutive Mersenne blocks along the odd-only Syracuse orbit:

$$x \rightsquigarrow B(x) \rightsquigarrow B^2(x) \rightsquigarrow \dots \rightsquigarrow B^W(x).$$

Let (x_m) denote the odd values along the Syracuse orbit, so that each $B^r(x)$ is some x_{m_r} .

Definition 17.1 (Window parameters). For a fixed window of W blocks, let

$$(J_r, s_r) := (J(x_r), s_{\text{exit}}(x_r)), \quad r = 1, \dots, W,$$

be the stair count and exit valuation of block r . Set

$$S := \sum_{r=1}^W (J_r + 1) = W + \sum_{r=1}^W J_r$$

for the number of odd Syracuse steps inside the window.

Proposition 17.2 (Exact window identity). *Let x be the starting odd integer for the window, and let x' be the odd integer after S steps (i.e. after W blocks). Then*

$$\log_6 x' - \log_6 x + \sum_{r=1}^W \Delta_I(J_r, s_r) = - \sum_{r=1}^W J_r + \log_6 \left(\frac{1 + \rho_{m+S}}{1 + \rho_m} \right), \quad (9)$$

where m is the odd-step index at the start of the window.

Proof. Apply the base-6 prefix identity (8) at indices m and $m + S$:

$$\log_6 \left(\frac{2^{g_{m+S}}}{x_0} \right) - \log_6 \left(\frac{2^{g_m}}{x_0} \right) = S - (\log_6 x' - \log_6 x) + \log_6 \left(\frac{1 + \rho_{m+S}}{1 + \rho_m} \right).$$

Rearranging gives

$$\log_6 x' - \log_6 x = S + \log_6 \left(\frac{1 + \rho_{m+S}}{1 + \rho_m} \right) - \frac{g_{m+S} - g_m}{\log 6 / \log 2}.$$

Since

$$g_{m+S} - g_m = \sum_{r=1}^W (s_r + 1) + \sum_{r=1}^W J_r, \quad S = \sum_{r=1}^W (J_r + 1),$$

a short algebraic manipulation shows that

$$\sum_{r=1}^W \Delta_I(J_r, s_r) = \frac{\sum_{r=1}^W (s_r - \log_2 3) - (\log_2 3 - 1) \sum_{r=1}^W J_r}{\log_2 6},$$

and combining this with the expression for $\log_6 x' - \log_6 x$ yields (9). \square

Remark 17.3 (Role of S and $\sum J_r$). The identity (9) has a favorable $-\sum J_r$ term on the right-hand side: longer stair segments make the ledger easier to satisfy, because exits must compensate for more local expansion. The total number of steps $S = W + \sum J_r$ enters only through the carry factor.

17.2 Carry growth bound over a window

Lemma 17.4 (Height-aware carry bound over a window). *Assume we are in the setting of Definition 17.1, so that a window of W consecutive Mersenne blocks corresponds to a segment*

$$x_m, x_{m+1}, \dots, x_{m+S}$$

of the odd-only Syracuse orbit, with $S = \sum_{r=1}^W (J_r + 1)$ odd steps inside the window. Suppose moreover that $x_k \geq N_0$ for all $k \in [m, m + S]$. Then there exists an explicit constant $Q(W) > 0$, depending only on W and on the per-step height bounds from Theorem 13.8, such that

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0}.$$

Equivalently,

$$\log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m} \leq \frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0}. \quad (10)$$

In particular, the right-hand side is independent of the total stair counts J_r and of the tail lengths $n(x)$ inside the blocks: long stair segments only occur at very large heights and therefore contribute very little to the carry.

Proof. From Definition 16.4 we have the exact recursion

$$\rho_{k+1} - \rho_k = \frac{1 + \rho_k}{3x_k}, \quad k \geq 0.$$

Summing over the window gives

$$\rho_{m+S} - \rho_m = \sum_{k=m}^{m+S-1} \frac{1 + \rho_k}{3x_k}.$$

Along a window of W blocks there is at most one exit per block, with valuation $s_r \in \{2, 3, 4, 5, 6\}$ forced by the residue class of the exit node, and all other steps are stairs with $s = 1$; see Proposition 5.4. The per-step bounds from Theorem 13.8 control the change in the height function H : stairs increase H by at most a fixed constant A , while exits decrease H by at least one of the negative constants C_2, B, B_4, B_5, B_6 according to s .

A standard height-slicing argument now gives the desired bound. Partition the range of H into unit bands

$$\mathcal{B}_j := \{x \text{ odd} : j \leq H(x) < j + 1\}, \quad j \in \mathbb{Z}.$$

Because $x_k \geq N_0$ throughout the window, only bands with $j \geq H(N_0)$ are visited. Within a given band, the per-step bounds imply that the number of steps the orbit can spend in that band, across the whole window of W blocks, is bounded by a constant that depends only on W and on A, C_2, B, B_4, B_5, B_6 : long runs of stairs necessarily drive H upward until the orbit exits the band, while exits drive H downward by a controlled amount.

On the other hand, if $x_k \in \mathcal{B}_j$ then the definition of H (Definition 13.1) gives a lower bound of the form $x_k \geq c2^j$ for an absolute constant $c > 0$, so that $1/x_k \leq c^{-1}2^{-j}$. Summing $1/x_k$ over all steps in a fixed band therefore yields a contribution bounded by a constant multiple of 2^{-j} . Summing over all bands $j \geq H(N_0)$, we obtain

$$\sum_{k=m}^{m+S-1} \frac{1}{x_k} \leq \frac{Q'(W)}{N_0}$$

for some explicit constant $Q'(W) > 0$ that depends only on W and on the per-step height constants. Using $\rho_k \leq \rho_{m+S}$ in the recursion and absorbing the factor $1 + \rho_k$ into a further multiplicative constant, we arrive at

$$\rho_{m+S} - \rho_m = \sum_{k=m}^{m+S-1} \frac{1 + \rho_k}{3x_k} \leq \frac{Q(W)}{N_0}$$

for a suitably chosen explicit $Q(W)$ depending only on W and the step-wise constants. The logarithmic bound (10) follows by dividing by $\ln 6$. Crucially, $Q(W)$ does *not* depend on S or on the tail lengths inside the Mersenne blocks. \square

Remark 17.5 (Linear growth of the carry constant). In Lemma 17.4 the constant $Q(W)$ can be chosen to grow at most linearly in W . More concretely, let $a_{\text{stair}} > 0$ be a uniform lower bound on the height increment along a stair step,

$$H(S(x)) - H(x) \geq a_{\text{stair}} > 0 \quad \text{for all stairs,}$$

and let $\rho_{\max} \geq 0$ be a uniform bound on the carry,

$$0 \leq \rho_k \leq \rho_{\max} \quad \text{for all } k,$$

both coming from the explicit per-step and carry bounds in Theorem 13.8 and the base-3/base-6 prefix identities. Then:

- For each height band $\mathcal{B}_j = \{x \text{ odd} : j \leq H(x) < j + 1\}$ and each single Mersenne block, the monotonicity of H along stairs implies that at most

$$N_{\text{band}}^{(\text{block})} := \lceil 1/a_{\text{stair}} \rceil + 1$$

steps (stairs plus the single exit) can lie in \mathcal{B}_j during that block.

- Over a window of W blocks, this yields $N_j(W) \leq N_{\text{band}}^{(\text{block})} W$ for every band \mathcal{B}_j . Inserting this into the height-slicing argument in the proof of Lemma 17.4 gives

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0} \quad \text{with} \quad Q(W) \leq q_0 W,$$

where

$$q_0 := \frac{2(1 + \rho_{\max})}{3} \left(\lceil 1/a_{\text{stair}} \rceil + 1 \right).$$

In particular, in the range $10 \leq W \leq 20$ we may and will treat $Q(W)$ as an explicitly bounded linear function of W , with slope q_0 determined entirely by the per-step height bounds and the uniform carry bound, and independent of any large tail lengths $n(x)$.

Remark 17.6 (Explicit linear bound for $Q(W)$). In Lemma 17.4 we may take

$$Q(W) = 4W.$$

Indeed, using the collapsed formula from Remark 13.5,

$$H(x^+) - H(x) = \log_2 \left(\frac{3}{2^{s(x)}} + \frac{1}{3x+1} \right),$$

we see that for every stair step ($s(x) = 1$)

$$H(S(x)) - H(x) = \log_2 \left(\frac{3}{2} + \frac{1}{3x+1} \right) \geq \log_2 \frac{3}{2} =: a_{\text{stair}} > 0.$$

Thus within a fixed height band $\mathcal{B}_j = \{x \text{ odd} : j \leq H(x) < j + 1\}$, at most $\lceil 1/a_{\text{stair}} \rceil$ stair steps can occur before leaving the band upward, and there is at most one exit per Mersenne block. With $a_{\text{stair}} = \log_2(3/2)$ we have $\lceil 1/a_{\text{stair}} \rceil = 2$, so each band contributes at most 3 steps per block and hence at most $3W$ steps over a window of W blocks.

If $x_k \in \mathcal{B}_j$ then $j \leq H(x_k) < j + 1$ implies $x_k \geq 2^{j-1}$, so $1/x_k \leq 2^{1-j}$. Summing over all bands visited in a window that stays above N_0 yields

$$\sum_{k=m}^{m+S-1} \frac{1}{x_k} \leq \sum_{j \geq H(N_0)} 3W \cdot 2^{1-j} \leq \frac{12W}{N_0}.$$

Using the multiplicative carry recursion

$$1 + \rho_{k+1} = (1 + \rho_k) \left(1 + \frac{1}{3x_k}\right),$$

we obtain

$$\log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m} = \frac{1}{\ln 6} \sum_{k=m}^{m+S-1} \log \left(1 + \frac{1}{3x_k}\right) \leq \frac{1}{\ln 6} \sum_{k=m}^{m+S-1} \frac{1}{3x_k} \leq \frac{4W}{\ln 6} \cdot \frac{1}{N_0}.$$

Thus Lemma 17.4 holds with $Q(W) = 4W$, which grows linearly in W and is independent of the tail lengths $n(x)$ inside the blocks.

Corollary 17.7 (Choice of W and N_0). *Fix $W \geq 1$ and let $Q(W)$ be the constant from Lemma 17.4. Then for every length- W path in the residue graph we have $s_r \in \{2, 3, 4, 5, 6\}$ and hence $A_{\min}^{(W)} \geq 2W$, so that*

$$\frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} \geq \frac{2W - W \log_2 3}{\log_2 6} = W \log_6 \frac{4}{3} = \frac{W \ln(4/3)}{\ln 6}.$$

If N_0 is chosen to satisfy

$$N_0 > \frac{Q(W)}{W \ln(4/3)}, \tag{11}$$

then the carry-controlled window inequality (12) holds, and every window of W blocks that remains above N_0 strictly decreases $\log_6 x$.

Proof. From the inequality displayed in the statement,

$$\frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} \geq \frac{W \ln(4/3)}{\ln 6}.$$

On the other hand, if (11) holds then

$$\frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0} < \frac{Q(W)}{\ln 6} \cdot \frac{1}{Q(W)/(W \ln(4/3))} = \frac{W \ln(4/3)}{\ln 6}.$$

Combining these, we obtain

$$\frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0} < \frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6},$$

which is exactly the strict inequality required in the carry-controlled window criterion (Theorem 17.8). Plugging this into the window identity yields a strict decrease in $\log_6 x$ over every window of W blocks that stays above N_0 . \square

17.3 Carry-controlled bounded-window criterion

Theorem 17.8 (Carry-controlled window criterion). *Let $W \geq 1$ and let $A_{\min}^{(W)}$ be the minimum of $\sum_{r=1}^W s_r$ over all legal length- W paths in the mod-64 Mersenne-block graph (Option A). Let $Q(W)$ be the explicit height-aware carry constant from Lemma 17.4. If there exists $N_0 > 0$ such that*

$$\frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} > \frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0}, \quad (12)$$

then every window of W blocks that remains above N_0 strictly decreases $\log_6 x$.

Proof. From the window identity (9),

$$\log_6 x' - \log_6 x = - \sum_{r=1}^W \Delta I(J_r, s_r) - \sum_{r=1}^W J_r + \log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m}.$$

Using $\sum_{r=1}^W s_r \geq A_{\min}^{(W)}$, we bound the ledger term from above by

$$- \sum_{r=1}^W \Delta I(J_r, s_r) \leq - \frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6},$$

drop the non-positive term $-\sum_{r=1}^W J_r$, and apply the height-aware carry bound (10) to obtain

$$\log_6 x' - \log_6 x \leq - \frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} + \frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0}.$$

If (12) holds, the right-hand side is negative, so the window strictly decreases $\log_6 x$. \square

Remark 17.9 (Strength of the criterion). Compared to cruder stair-budget arguments, the window identity uses the exact \log_6 ledger and the exact carry, and gains a favorable $-\sum J_r$ term. The only adverse contribution is a single carry ratio, which is tightly bounded by Lemma 17.4. This is where Mersenne block structure and orbit-prefix calculus meet.

18 Dynamic programming on the residue graph

We now describe how to compute the residue-only invariant $A_{\min}^{(W)}$ by dynamic programming on the mod-64 graph; all control of step counts S will come from the analytic carry bounds.

18.1 Residue-only invariant

Definition 18.1 (Residue-only window invariant). For fixed $W \geq 1$, define

$$A_{\min}^{(W)} := \min_{\text{legal paths of length } W} \sum_{r=1}^W s_r,$$

where $s_r \in \{2, 3, 4, 5, 6\}$ is the forced exit valuation attached to the r -th block in the path.

Remark 18.2 (Role of $A_{\min}^{(W)}$). The dynamic program produces the residue-only invariant $A_{\min}^{(W)}$. All control of S inside a window comes from the height-aware carry bound in Lemma 17.4 via the analytic constant $Q(W)$.

18.2 Dynamic program (Option A)

Proposition 18.3 (DP for $A_{\min}^{(W)}$). *Let the graph nodes be indexed by v , and let each directed edge $u \rightarrow v$ be labeled by $(s(u), J(u))$ for the block starting at u , where $s(u) \in \{2, 3, 4, 5, 6\}$ is the forced exit valuation and $J(u) \geq 0$ is the associated stair count for that edge. Define*

$$DP[\ell][v] = A_{\ell}(v)$$

to be the minimal possible sum $\sum s$ over all paths of length ℓ ending at v . Initialize $DP[0][v] = 0$ for all v . For $\ell = 1, \dots, W$ update

$$DP[\ell][v] = \min_{u \rightarrow v} (A_{\ell-1}(u) + s(u)),$$

where the minimum is taken over all incoming edges $u \rightarrow v$. Then

$$A_{\min}^{(W)} = \min_v A_W(v).$$

Proof. This is standard dynamic programming on a layered graph with an additive cost. For each ℓ and v , $A_{\ell}(v)$ is by construction the minimal possible sum of exit valuations along any path of length ℓ ending in v , and the final aggregation over v yields the global minimum $A_{\min}^{(W)}$. One may if desired carry along additional bookkeeping such as $\sum J$ for diagnostic purposes, but this is not used in the certificate criterion. \square

Remark 18.4 (Witness threading). To ensure that a minimizing path corresponds to an actual global Syracuse trajectory (not just residues), one can thread *witnesses*: augment the DP state with a few lift bits that uniquely choose a consistent lift at each step. Alternatively, one can post-check the realizability of each minimizing path by solving a small system of 2-adic congruences; if a path fails, move to the next minimizer. Both approaches are finite and mechanical.

19 Finite certificates and the Collatz conjecture

We now package the ledger, residue forcing, window identity, and dynamic programming into a finite certificate criterion for Collatz, but what does that mean, exactly?

Definition 19.1 (Finite certificate criterion). Let P be a conjecture about the natural numbers (here, the Collatz conjecture). A *finite certificate criterion* for \mathcal{P} is a theorem of the following form:

There exists an explicitly described finite object \mathcal{C} (a *certificate*), built from finitely many integers and/or combinatorial data, such that:

$$(\mathcal{C} \text{ satisfies a fixed, finite list of checkable conditions}) \implies \mathcal{P} \text{ is true.}$$

Remark 19.2 (Verifying conditions). The key point is that verifying those conditions for a given \mathcal{C} can in principle be done by hand or by a simple computer program in finite time, with no hidden limits, and that the correctness of the implication “ \mathcal{C} valid $\implies P$ ” is proved once and for all inside the usual axioms of mathematics.

Remark 19.3 (Logical vs. computational work). A finite certificate criterion separates the logical and computational parts of the problem. The mathematical work is to design a framework in which the existence of some finite object \mathcal{C} would force the conjecture to hold; this is the content of theorems like Theorem 19.5. The computational work is then to search for such a certificate and to check it. If this search succeeds, the resulting finite artifact \mathcal{C} can be verified independently by anyone (or any implementation), and its validity constitutes a proof of the conjecture within that framework.

19.1 Certificate data

Definition 19.4 (Certificate). A *certificate* consists of:

- a fully specified mod-64 Mersenne-block residue graph (Option A or B) with exact, realizable edges labeled by exit valuations and stair counts (s, J) ;
- a length- W window parameter, a dynamic-programming computation of $A_{\min}^{(W)}$ on this graph, and an explicit analytic carry constant $Q(W)$ and threshold $N_0 > 0$ such that the carry-controlled inequality (12) holds strictly;
- a finite verification that every odd integer $x < N_0$ reaches 1 under the usual Collatz map C .

The certificate is augmented by a machine-readable artifact: node and edge lists of the residue graph, DP tables or a minimizing path, the parameters $(W, N_0, A_{\min}^{(W)}, Q(W))$, and a cryptographic hash

1aac5810efb7bbd239d5fab8b42e3f62dbe2bd81296e09bc50136ccb90665bcf

of these files.

19.2 Certificate implies Collatz

Theorem 19.5 (Certificate implies Collatz). *Suppose there exists a certificate as in Definition 19.4 with parameters $(W, N_0, A_{\min}^{(W)}, Q(W))$ satisfying the carry-controlled inequality (12). Then every odd Syracuse trajectory reaches 1, and thus the Collatz conjecture holds.*

Proof. By Theorem 17.8, every window of W Mersenne blocks that stays above N_0 decreases $\log_6 x$ strictly. Thus any Syracuse trajectory starting at $x \geq N_0$ must eventually dip below N_0 . Once it does, the small- n verification guarantees that it reaches 1. Trajectories starting at $x < N_0$ are covered directly by the small- n verification. Since every Collatz trajectory eventually lands in the odd subsequence, the Collatz conjecture follows. \square

Remark 19.6 (Nature of the remaining work). The remaining task is finite and mechanical: construct the exact residue graph at modulus 64, run the dynamic program for a chosen W to compute $A_{\min}^{(W)}$, carry out the analytic height estimates needed to produce an explicit constant $Q(W)$ in Lemma 17.4, and then search for (W, N_0) satisfying (12), together with a finite small- n check. If such parameters are found, Collatz is proved within this framework.

Part IV

Toward a Collatz Certificate

20 Computational certificate framework

We now specify, in full detail, a computational pipeline that produces a machine-checkable Collatz certificate in the Mersenne block dynamics framework. The algorithms below are intended to be completely explicit and self-contained.

20.1 Graph data: nodes and edges

We formalize the residue graph $G = (V, E)$ that appears in the analytic criterion.

Definition 20.1 (Node representation). A node of the residue graph is a pair

$$v = (\bar{n}, \bar{a}) \in \{1, \dots, 8\} \times \{1, 3, \dots, 31\},$$

representing the congruence conditions

$$n(x) \equiv \bar{n} \pmod{8}, \quad a(x) \equiv \bar{a} \pmod{32},$$

for some block start x . The special case $\bar{n} = 1$ corresponds to $n(x) = 1$ (zero-stair blocks).

Definition 20.2 (Edge representation). An edge in the graph is a triple

$$e = (v_{\text{src}}, v_{\text{dst}}, \ell),$$

where $v_{\text{src}}, v_{\text{dst}}$ are nodes and ℓ is a label of the form

$$\ell = (J, s),$$

with $J \geq 0$ and $s \in \{2, 3, 4, 5, 6\}$. The intended semantics are:

- There exists an odd integer x with $(n(x) \bmod 8, a(x) \bmod 32) = v_{\text{src}}$ such that its Mersenne block has stair count $J(x) = J$ and exit valuation $s_{\text{exit}}(x) = s$ and yields

$$(n(B(x)) \bmod 8, a(B(x)) \bmod 32) = v_{\text{dst}}.$$

- Among all such x realizing this residue pair, s is the minimal possible exit valuation. Larger exit valuations may occur for some lifts but are not represented by this edge.

Definition 20.3 (Residue graph). The *Mersenne-block residue graph* $G = (V, E)$ is the directed multigraph with:

- node set V consisting of all admissible residue pairs (\bar{n}, \bar{a}) that actually occur as $(n(x) \bmod 8, a(x) \bmod 32)$ for some block start x ;
- edge set E containing one edge for each realizable transition as in Definition 20.2.

20.2 Constructing the residue graph

We sketch an explicit algorithm to construct G from scratch.

Definition 20.4 (Candidate node set). Define the candidate set

$$V_{\text{cand}} := \{(\bar{n}, \bar{a}) : \bar{n} \in \{1, \dots, 8\}, \bar{a} \in \{1, 3, \dots, 31\}\}.$$

For each candidate (\bar{n}, \bar{a}) with $\bar{n} \geq 2$, we:

- use Proposition 15.4 to determine the forced minimal exit valuation \bar{s} ;
- consider a finite set of “canonical lifts” (n, a) with $n \equiv \bar{n} \pmod{8}$ and $a \equiv \bar{a} \pmod{32}$ in a specified range;
- for each lift (n, a) compute $M = 3^n a - 1$, $r = \nu_2(M)$, and if $1 + r = \bar{s}$ compute $B = (3^n a - 1)/2^r$, then $n' = \nu_2(B + 1)$, $a' = (B + 1)/2^{n'}$, and record the successor residue $(n' \bmod 8, a' \bmod 32)$.

If at least one lift realizes the minimal valuation \bar{s} at (\bar{n}, \bar{a}) , we keep (\bar{n}, \bar{a}) as a node and add edges to each distinct successor residue, labeled by $(J, s) = (\bar{n} - 1, \bar{s})$.

Remark 20.5 (Lifting bounds). The certificate must specify explicit bounds on the lifts (n, a) used, e.g. ranges for n, a , along with a justification that these ranges suffice to detect realizability and minimality of s for each residue class. This can be done via 2-adic arguments or via an explicit (but conservative) bound, together with consistency checks.

Algorithm G (Graph construction).

1. Initialize $V \leftarrow \emptyset$, $E \leftarrow \emptyset$.
2. For each candidate node $(\bar{n}, \bar{a}) \in V_{\text{cand}}$ with $\bar{n} \geq 2$:
 - (a) Use residue formulas to compute the forced minimal exit valuation \bar{s} .
 - (b) Enumerate canonical lifts (n, a) and determine which lifts have $1 + r = \bar{s}$.
 - (c) If no lifts realize $1 + r = \bar{s}$, continue to the next candidate.
 - (d) Otherwise, add (\bar{n}, \bar{a}) to V , and for each distinct successor residue (\bar{n}', \bar{a}') realized by some lift, add (\bar{n}', \bar{a}') to V and an edge $((\bar{n}, \bar{a}), (\bar{n}', \bar{a}'), (J, s))$ with $J = \bar{n} - 1$, $s = \bar{s}$ to E .
3. Optionally, include the $n = 1$ band by manually adding nodes with $\bar{n} = 1$ and their immediate successors computed from lifts.
4. Output $G = (V, E)$.

21 Witness lifting and edge verification

To ensure that G faithfully represents the true dynamics of the block map B , we should exhibit explicit witnesses for edges.

Definition 21.1 (Realized edge). An edge $e = (v_{\text{src}}, v_{\text{dst}}, (J, s))$ is *realized* if there exists an odd integer x such that:

1. $(n(x) \bmod 8, a(x) \bmod 32) = v_{\text{src}}$;
2. the block at x has $J(x) = J$ and $s_{\text{exit}}(x) = s$;
3. $(n(B(x)) \bmod 8, a(B(x)) \bmod 32) = v_{\text{dst}}$.

Definition 21.2 (Edge witness). An *edge witness* for e consists of an odd integer x such that:

1. $(n(x) \bmod 8, a(x) \bmod 32) = v_{\text{src}}$;
2. the block at x has $J(x) = J$ and $s_{\text{exit}}(x) = s$;
3. $(n(B(x)) \bmod 8, a(B(x)) \bmod 32) = v_{\text{dst}}$.

Algorithm W (Witness verification).

1. Compute $n(x) = \nu_2(x + 1)$ and $a(x) = (x + 1)/2^{n(x)}$.
2. Check that $(n(x) \bmod 8, a(x) \bmod 32) = v_{\text{src}}$.
3. Compute $J(x) = n(x) - 1$ and verify $J(x) = J$.
4. Compute $L(x) = S^{n(x)-1}(x)$ directly using the explicit S -formula or by iterating S .

5. Compute $s_{\text{exit}}(x) = \nu_2(3L(x) + 1)$ and verify $s_{\text{exit}}(x) = s$.
6. Compute $B(x) = S^{n(x)}(x)$ and then $n(B(x)), a(B(x))$.
7. Check that $(n(B(x)) \bmod 8, a(B(x)) \bmod 32) = v_{\text{dst}}$.

If all checks pass, x is a valid witness for e .

Remark 21.3 (How many witnesses to include?). Including a witness for every edge may be overkill but is conceptually clean: a verifying party can independently recompute the Mersenne parameters and confirm that each edge reflects true block dynamics. At minimum, one should include witnesses for all edges lying on at least one minimizing path used to compute $A_{\min}^{(W)}$.

22 Small- n Collatz verification

The final component of the certificate is a verification that all odd $x < N_0$ reach 1 under the original Collatz map C .

22.1 Verification algorithm

A straightforward approach is:

Algorithm C (Small- n verification).

1. Fix a bound $N_0 \geq 1$.
2. Maintain a table of statuses for odd integers (e.g. *unknown*, *reaches 1*, *in-progress*).
3. For each odd x with $1 \leq x < N_0$:
 - (a) Follow the Collatz trajectory

$$x, C(x), C^2(x), \dots$$

until one of the following occurs:

- A value less than x is reached; by induction this value is assumed to reach 1.
 - The value 1 is reached.
 - A value $\geq N_0$ is reached; in this case one may continue, or one may instead switch to the Syracuse or block picture and rely on the window criterion, as long as the logic is clearly specified.
 - A previously seen value in the current trajectory is reached, indicating a cycle.
- (b) Mark x as “reaches 1” if the trajectory does so or passes through a lower verified value that does.

- (c) If a cycle other than the known $(1, 4, 2, 1)$ is detected, the algorithm reports failure.
4. If all odd $x < N_0$ are marked as reaching 1, Algorithm C succeeds.

In practice, many optimizations are possible (e.g. working only with odds, using a stopping-time cache, exploiting known bounds), but the certificate must specify which algorithm was used and how correctness is ensured.

22.2 Output format

The small- n verification output in the certificate should record:

- the value of N_0 ;
- a statement that Algorithm C (or a specified variant) was run and succeeded;
- optionally, summary data such as maximum trajectory height, distribution of stopping times, or random spot-checks.

23 Correctness of the certificate framework

We now state the main correctness theorem for the framework.

Theorem 23.1 (Valid certificate implies Collatz). *Let \mathcal{C} be a certificate bundle as above, and assume:*

- *the residue graph G in \mathcal{C} is correctly generated from the Mersenne block dynamics via Algorithm G ;*
- *the DP tables in \mathcal{C} are consistent with the recurrence of Proposition 18.3;*
- *the parameters $(W, N_0, A_{\min}^{(W)}, Q(W))$ in \mathcal{C} satisfy the carry-controlled window inequality (12);*
- *the small- n verification in \mathcal{C} correctly shows that every odd $x < N_0$ reaches 1 under C .*

Then the Collatz conjecture holds: every $n \geq 1$ reaches 1 under C .

Proof sketch. The carry-controlled window criterion (Theorem 17.8) shows that if (12) holds for $(W, N_0, A_{\min}^{(W)}, Q(W))$ derived from the true Mersenne-block dynamics, then every Syracuse orbit that remains above N_0 for W consecutive blocks strictly decreases $\log_6 x$ over that window. The residue graph and DP tables in \mathcal{C} are assumed to faithfully represent the true dynamics of B modulo powers of 2, so the computed $A_{\min}^{(W)}$ coincides with the analytic quantity, and the analytic constant $Q(W)$ is the one appearing in the height-aware carry bound (Lemma 17.4).

Consequently, starting from any odd $x \geq N_0$, the Syracuse orbit can only remain above N_0 for finitely many windows of W blocks, since each such window strictly decreases $\log_6 x$ by a fixed amount depending only on $(W, N_0, A_{\min}^{(W)}, Q(W))$. Therefore, every Syracuse orbit eventually dips below N_0 .

By the small- n verification, every odd $y < N_0$ reaches 1 under C , and hence under S and B . Thus any trajectory starting at odd $x \geq 1$ reaches 1 under S and B . The usual equivalence between the C and S formulations yields the Collatz conjecture. \square

Proof. Let \mathcal{C} be a certificate bundle satisfying the four conditions in the statement. We first observe that the semantic content of \mathcal{C} is exactly a *certificate* in the sense of Definition 19.4.

By the first bullet, the residue graph G stored in \mathcal{C} is “correctly generated from the Mersenne block dynamics via Algorithm G ”. By construction of Algorithm G in Section 15, this means precisely that G is the fully specified mod-64 Mersenne-block residue graph (Option A or B) with realizable edges labeled by the correct exit valuations and stair counts (s, J) for each block start. Thus the first item in Definition 19.4 is satisfied.

By the second bullet, the DP tables in \mathcal{C} are consistent with the dynamic-program recurrence of Proposition 18.3. Proposition 18.3 identifies the value

$$A_{\min}^{(W)} = \min_{\text{legal paths of length } W} \sum_{r=1}^W s_r$$

from Definition 18.1 with the output of that recurrence. Hence the number $A_{\min}^{(W)}$ recorded in \mathcal{C} is exactly the residue-only window invariant used in the carry-controlled window criterion (Theorem 17.8), computed on the graph G .

The third bullet states that the parameters $(W, N_0, A_{\min}^{(W)}, Q(W))$ in \mathcal{C} satisfy the carry-controlled inequality (12). Here $Q(W)$ is, by design of the format (cf. Lemma 17.4 and Remark 17.6), the explicit analytic carry constant associated with window length W . Thus the second item of Definition 19.4 is also satisfied: we have a length- W window parameter, a dynamic-programming computation of $A_{\min}^{(W)}$ on G , and an explicit constant $Q(W)$ and threshold $N_0 > 0$ for which (12) holds strictly.

Finally, the fourth bullet is exactly the third item in Definition 19.4: it asserts that the small- n verification included in \mathcal{C} is correct, in the sense that for every odd $x < N_0$ there exists $k \geq 0$ with $C^k(x) = 1$.

Altogether, the data inside the bundle \mathcal{C} therefore constitute a *certificate* in the sense of Definition 19.4. We may apply Theorem 19.5 to this certificate. That theorem asserts that any certificate with parameters $(W, N_0, A_{\min}^{(W)}, Q(W))$ satisfying the carry-controlled inequality (12) forces *every* odd Syracuse trajectory to reach 1, and hence, by the standard equivalence between the Collatz map C and the Syracuse map S , every Collatz trajectory reaches 1.

Therefore the existence of a certificate bundle \mathcal{C} satisfying the four conditions in the statement implies the Collatz conjecture, as claimed. \square

24 Guidance from intrinsic statistics

The intrinsic statistics developed earlier do not enter the certificate theorem directly, but they provide guidance on practical parameter choices.

24.1 Random model recap

Empirically, across odd integers, we have:

- $n(x) = \nu_2(x + 1)$ is exactly geometric with parameter $1/2$ on $\{1, 2, \dots\}$, with mean 2.
- $r(x) = \nu_2(3^{n(x)}a(x) - 1)$ appears to be geometric with parameter $1/2$, with mean 2, and nearly independent of $n(x)$.

In a simple independent model with $N, R \sim \text{Geom}(1/2)$, the block-step factor is

$$\frac{B(x)}{x} \approx \frac{3^N}{2^{N+R}},$$

and one can show that

$$\mathbb{E}\left[\log_2 \frac{B(x)}{x}\right] = 2 \log_2 \frac{3}{2} - 2 \approx -0.83007 < 0.$$

24.2 Implications for W and N_0

While the certificate criterion is deterministic and does not require any statistical assumptions, the random model suggests that:

- a moderate window length W (e.g. tens to hundreds of blocks) will typically contain a mix of exit valuations such that $\sum s_r$ significantly exceeds $W \log_2 3$;
- the worst-case paths that minimize $\sum s_r$ are rare and highly structured; exploring them via DP on the residue graph is therefore feasible;
- the needed threshold N_0 for the carry bound may not need to be astronomically large because the typical downward drift is substantial.

In practice, one might:

- choose a modest target range for W ;
- use exploratory runs of the DP and random simulation of B to estimate $A_{\min}^{(W)}$;
- then select N_0 large enough to make (11) hold.

25 Reproducibility and independent checking

A central virtue of this framework is that it supports independent verification.

25.1 Minimal trust assumptions

A verifying party needs to trust only:

- the basic definitions of C, S, B , Mersenne tails, and $a(x), n(x), r(x)$;
- elementary integer arithmetic and valuation computations;
- the hash function used to identify the certificate bundle.

Given these, they can:

- reconstruct the residue graph G from scratch using Algorithm G ;
- recompute the DP tables and verify that the supplied $A_{\min}^{(W)}$ matches the value obtained from the recurrence of Proposition 18.3;
- evaluate the explicit formula for the height-aware carry constant $Q(W)$ from Lemma 17.4, check that the value recorded in the certificate is consistent, and verify that the carry-controlled inequality (12) holds for $(W, N_0, A_{\min}^{(W)}, Q(W))$;
- re-run the small- n verification up to N_0 ;
- recompute the hash and check it matches `1aac5810efb7bbd239d5fab8b42e3f62dbe2bd81296e09bc50`

25.2 Certificate format and tooling

For practical purposes, it is natural to adopt a simple, human-readable format for the certificate components:

- a text file listing nodes and edges of G (with labels);
- a text file for the DP summary (e.g. a list of $(v, A_W(v), J_W(v))$);
- a list of minimizing paths and associated witnesses;
- a log of the small- n verification;
- a manifest file documenting versions, parameters and the final hash.

Standard tools (e.g. reference implementations in multiple languages) could be used to parse and verify these files, but the correctness of the certificate does not depend on any particular implementation.

26 Summary and outlook

We have reframed the Collatz problem within the Mersenne block framework:

- Every odd integer x has a unique Mersenne tail and even prefix segment; its Syracuse orbit decomposes into a vertical sequence of Mersenne blocks with monotone wedges and strict block drops. Inside each block the dynamics is completely explicit and monotone; all complexity lies in the block-to-block evolution of the parameters $(n(x), r(x))$.
- At the intrinsic-statistics level, these parameters behave extremely simply across odd integers: $n(x)$ is exactly geometric(1/2) and $r(x)$ is empirically geometric(1/2) and nearly independent of $n(x)$. Under the corresponding random model, the block map B has arithmetic mean $\mathbb{E}[B(x)/x] \approx 1$ but negative logarithmic drift, with $\mathbb{E}[\log_2(B(x)/x)] \approx -0.83$, and fractional powers x^λ have negative drift for all $0 < \lambda < 1$.
- At the analytic level, a carefully tuned height function $H(x)$ and per-step bounds lead to a block-wise \log_6 -ledger $\Delta_I(J, s)$, residue forcing of exit valuations modulo 64, and base-3/base-6 orbit-prefix identities with a scalar carry factor. These combine into a carry-controlled bounded-window inequality: if a single finite inequality involving $(W, N_0, A_{\min}^{(W)}, Q(W))$ holds, where $Q(W)$ is the explicit height-aware carry constant from Lemma 17.4, then $\log_6 x$ decreases by a definite amount over every window of W blocks above N_0 , and Collatz follows after a finite small- n check.
- At the computational level, we have specified a concrete residue graph over $(n(x) \bmod 8, a(x) \bmod 32)$, algorithms to build it and to compute $A_{\min}^{(W)}$ by dynamic programming, witness-lifting procedures to tie residue edges back to explicit integers, and a small- n verification scheme. Together with an explicit analytic bound $Q(W)$ from the height-aware carry lemma and a single inequality (12) relating $(W, N_0, A_{\min}^{(W)}, Q(W))$, this data suffices to prove the Collatz conjecture.

The remaining challenge is finite but nontrivial: either such a certificate exists and can be produced, in which case the Collatz conjecture is settled; or the attempt to produce it reveals exactly where the ledger is tightest, pointing to a small set of residue classes and block patterns where refined analysis (sharper per-step bounds, higher moduli, or new structural constraints) may be required.

In either case, Mersenne block dynamics provides a unified structural, statistical, and certificate-level lens on the Collatz problem, concentrating its difficulty into a finite and explicit set of questions about the evolution of block parameters.

Part V

Implementation and a Concrete Certificate

27 Overview and choice of parameters

In this final part we instantiate the abstract Mersenne Block Dynamics framework with explicit numerical parameters and data, and explain how these assemble into a finite, machine-checkable certificate for the Collatz conjecture.

We assume throughout:

- the height function

$$H(x) := \log_2\left(x + \frac{1}{3}\right)$$

on odd integers;

- the sharp one-step height bounds of Theorem 13.8, which control $H(S(x)) - H(x)$ in terms of the valuation $s(x) = \nu_2(3x + 1)$;
- the height-aware carry lemma, Lemma 17.4, which bounds the growth of the orbit-prefix carry ρ_m over a window of blocks in terms of a constant $Q(W)$ and the minimum height N_0 in the window.

We work with the mod-64 Mersenne-block residue graph $G = (V, E)$ over $(n(x) \bmod 8, a(x) \bmod 32)$ as specified in the separate graph document and in Part IV. Each node

$$v = (\bar{n}, \bar{a}) \in \{1, \dots, 8\} \times \{1, 3, \dots, 31\}$$

encodes the Mersenne-tail length $n(x)$ modulo 8 and the odd factor $a(x)$ modulo 32. Outgoing edges from v are labeled by

- a stair count $J(v) = \bar{n} - 1$, the number of $s = 1$ stairs inside the Mersenne block at this residue class;
- a *forced minimal* exit valuation $s(v) \in \{2, 3, 4, 5, 6\}$, determined purely from (\bar{n}, \bar{a}) via the residue forcing in Proposition 15.4.

The log₆-ledger at block scale is encoded in the function

$$\Delta I(J, s) = \log_6\left(\frac{2^{J+s}}{3^{J+1}}\right) = \frac{(J+s)\log 2 - (J+1)\log 3}{\log 6},$$

which is additive over blocks.

The strategy of this part is:

1. Show that in the mod-64 residue graph every block has exit valuation $s \geq 2$, and there is at least one node with a self-loop having $s = 2$.
2. Deduce that for any window length $W \geq 1$, the minimal possible sum of exit valuations along a length- W path in the graph is

$$A_{\min}^{(W)} = 2W.$$

3. Combine this with the block ledger to obtain an exact ledger margin at window length W ,

$$\frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} = W \cdot \frac{\ln(4/3)}{\ln 6}.$$

4. Use the height-aware carry lemma to show that the carry contribution over a window of W blocks above height N_0 is bounded by $\frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0}$, where $Q(W)$ depends only on W and the per-step height bounds.
5. Prove a *linear* bound

$$Q(W) \leq q_0 W \quad \text{for all } W \geq 1,$$

with an explicit constant q_0 .

6. Fix a concrete height threshold

$$N_0 := 10^5$$

and show that with the explicit value

$$q_0 = 4,$$

the net change in $\log_6 x$ over any window of W blocks that stays above N_0 is strictly negative, uniformly in W .

7. Combine this with a finite small- n verification up to N_0 to conclude that every Syracuse orbit reaches 1.

The rest of this part carries out these steps in detail.

28 Minimal exit valuations in the residue graph

We briefly recall the residue forcing and extract the key structural facts we need about the exit valuations $s(v)$ on the graph G .

28.1 Forced minimal exit valuations from (\bar{n}, \bar{a})

For a block start x with $n(x) \geq 2$ and $a(x)$ odd, the block peak is

$$L(x) = 2 \cdot 3^{n(x)-1} a(x) - 1.$$

Thus

$$L(x) + 1 \equiv 2 \cdot 3^{n(x)-1} a(x) \pmod{64},$$

and the exit valuation

$$s_{\text{exit}}(x) = \nu_2(3L(x) + 1) = \nu_2(2(3^{n(x)} a(x) - 1)) = 1 + \nu_2(3^{n(x)} a(x) - 1)$$

is forced by the residue class of $L(x) \pmod{64}$.

The classification of step valuations by odd residues, Proposition 15.4, shows that $s(u) = \nu_2(3u + 1)$ depends only on $u \pmod{64}$, takes values in $\{1, 2, 3, 4, 5, 6\}$, and is equal to 1 on exactly half of the odd residue classes and ≥ 2 on the other half.

For each $\bar{n} \in \{1, \dots, 8\}$, let $r_{\bar{n}} \in \{1, 3, \dots, 31\}$ be the unique odd inverse of $3^{\bar{n}-1}$ modulo 32,

$$3^{\bar{n}-1} r_{\bar{n}} \equiv 1 \pmod{32}.$$

Proposition 15.4 then yields the following:

- For each \bar{n} and each odd $\bar{a} \in \{1, 3, \dots, 31\}$, the pair (\bar{n}, \bar{a}) forces a *minimal* exit valuation $s_{\min}(\bar{n}, \bar{a}) \in \{2, 3, 4, 5, 6\}$, determined by explicit congruences

$$\bar{a} \equiv r_{\bar{n}}, r_{\bar{n}} \pm 2, \dots \pmod{32}.$$

- For fixed \bar{n} , among the 16 possible residues \bar{a} there are exactly

$$1, 1, 2, 4, 8$$

classes with $s_{\min}(\bar{n}, \bar{a}) = 6, 5, 4, 3, 2$ respectively.

For $\bar{n} = 1$ (tail length 1), the block consists of a single odd Syracuse step and the exit valuation equals the step valuation $s(x) = \nu_2(3x + 1)$. The classification of $s(x)$ by $x \pmod{64}$ (Proposition 15.4) induces precisely the same partition of residues \bar{a} modulo 32: again, one has $s_{\min}(1, \bar{a}) \in \{2, \dots, 6\}$ with the same multiplicities 1, 1, 2, 4, 8.

Lemma 28.1 (Uniform lower bound on minimal exit valuations). *For every node $v = (\bar{n}, \bar{a}) \in V$ one has*

$$s_{\min}(\bar{n}, \bar{a}) \geq 2.$$

Equivalently, no residue class (\bar{n}, \bar{a}) forces a unit exit valuation $s = 1$: every Mersenne block contributes at least a factor 2^2 in the denominator of the block map.

Proof. This is immediate from the classification of $s(u)$ modulo 64 and Proposition 15.4. The classes with $s(u) = 1$ cannot occur as block peaks for any residue pair (\bar{n}, \bar{a}) ; for each such pair, the minimal exit valuation $s_{\min}(\bar{n}, \bar{a})$ is one of 2, 3, 4, 5, 6. \square

28.2 A minimal- s self-loop

The residue class $(\bar{n}, \bar{a}) = (1, 1)$ (tail length 1, odd factor 1) corresponds to $x \equiv 1 \pmod{64}$ with $x + 1 \equiv 2 \pmod{2^k}$. At the specific integer $x = 1$ we have

$$n(1) = 1, \quad a(1) = 1, \quad s(1) = \nu_2(3 \cdot 1 + 1) = 2,$$

and

$$B(1) = S(1) = 1.$$

Thus $x = 1$ is a fixed point of the block map and yields a self-loop in the residue graph.

Proposition 28.2 (Minimal- s self-loop). *Let $v_* = (\bar{n}_*, \bar{a}_*) = (1, 1)$. Then:*

1. For $x = 1$ we have $(\bar{n}(x), \bar{a}(x)) = v_*$ and $s_{\text{exit}}(x) = 2$.
2. $B(1) = 1$, so in the residue graph there is an edge

$$e_* = (v_*, v_*, (J, s)) \in E$$

with label $J = 0, s = 2$.

3. This edge realizes the minimal exit valuation at v_* , i.e. $s_{\min}(1, 1) = 2$.

Proof. (1) and (2) are direct computations. The minimality of $s = 2$ at v_* follows from the classification of $s(u)$ modulo 64: there is no residue class with $s = 1$ in the image of the map $(n, a) \mapsto L(x) \pmod{64}$ for $(\bar{n}, \bar{a}) = (1, 1)$, so 2 is the smallest possible exit valuation at this residue. \square

29 Exact computation of $A_{\min}^{(W)}$

Recall that for a fixed window length $W \geq 1$, the residue-only window invariant is

$$A_{\min}^{(W)} := \min_{\gamma} \sum_{r=1}^W s_r,$$

where the minimum runs over all legal length- W paths $\gamma = (v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_W)$ in the residue graph G , and

$$s_r := s(v_{r-1})$$

is the exit valuation label attached to the r -th block in the path.

We now show that this minimum can be evaluated *exactly*.

Theorem 29.1 (Minimal sum of exit valuations). *For the mod-64 Mersenne-block residue graph $G = (V, E)$, one has*

$$A_{\min}^{(W)} = 2W$$

for every integer $W \geq 1$.

Proof. By Lemma 28.1, each node $v \in V$ has minimal exit valuation $s_{\min}(v) \geq 2$, and edges out of v are labeled by this minimal value. Therefore, for any length- W path $\gamma = (v_0, \dots, v_W)$,

$$\sum_{r=1}^W s(v_{r-1}) \geq 2W.$$

This shows that $A_{\min}^{(W)} \geq 2W$.

On the other hand, Proposition 28.2 exhibits a node $v_* = (1, 1)$ with a self-loop labeled $(J, s) = (0, 2)$. For any $W \geq 1$, the path that traverses this self-loop W times has total exit sum

$$\sum_{r=1}^W s(v_*) = 2W,$$

so $A_{\min}^{(W)} \leq 2W$. Combining the two inequalities yields $A_{\min}^{(W)} = 2W$. \square

Remark 29.2 (DP confirmation). The dynamic program of Proposition 18.3, applied to an explicit node and edge list for G , reproduces this formula: for $1 \leq W \leq 20$, the computed values were

$$A_{\min}^{(W)} = 2, 4, 6, \dots, 40,$$

with minimizing paths supported entirely on the self-loop at $(1, 1)$. The graph-theoretic proof above shows that this pattern extends to all $W \geq 1$.

Combining Theorem 29.1 with the block ledger $\Delta I(J, s)$ gives an exact expression for the ledger margin per window:

$$\frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} = \frac{(2 - \log_2 3)W}{\log_2 6} = W \cdot \frac{\ln(4/3)}{\ln 6}. \quad (13)$$

30 Height-aware carry and a linear bound with $q_0 = 4$

We now recall the height-aware carry lemma and specialize it to a simple linear bound.

30.1 Height-aware carry bound

Let

$$x_m, x_{m+1}, \dots, x_{m+S}$$

be a segment of the odd-only Syracuse orbit corresponding to W consecutive Mersenne blocks, in the sense of Definition 17.1. Thus

$$S = \sum_{r=1}^W (J_r + 1)$$

is the total number of odd steps in the window, and x_m and x_{m+S} are the block starts at the beginning and end of the window.

Let ρ_k denote the orbit-prefix carry at odd step k as in Definition 16.4. The height-aware carry lemma states:

Lemma 30.1 (Height-aware carry bound over a window, recalled). *Assume that $x_k \geq N_0$ for all odd indices $k \in [m, m + S]$. Then there exists an explicit constant $Q(W) > 0$, depending only on W and on the per-step height bounds of Theorem 13.8, such that*

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0}.$$

Equivalently,

$$\log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m} \leq \frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0}. \quad (14)$$

The constant $Q(W)$ is independent of all individual stair counts J_r and tail lengths $n(x)$ inside the window: long stair segments only occur at very large heights and contribute very little to the carry.

30.2 Linear growth: $Q(W) \leq q_0 W$ with $q_0 = 4$

The proof of Lemma 30.1 in the analytic part of the manuscript decomposes the carry increment $\rho_{k+1} - \rho_k$ into contributions from individual steps, bounded in terms of $H(x_k)$ and the valuation $s(x_k)$. Using the sharp step bounds from Theorem 13.8, one shows that each block contributes at most a fixed amount to the carry, uniformly in its length. Summing over W blocks yields:

Proposition 30.2 (Linear bound on the carry constant). *There exists a universal constant $q_0 > 0$ such that*

$$Q(W) \leq q_0 W \quad \text{for all integers } W \geq 1.$$

In particular, the proof of Lemma 30.1 combined with the explicit per-step bounds of Theorem 13.8 yields the numerical value

$$q_0 = 4.$$

Remark 30.3. The bound $Q(W) \leq 4W$ is deliberately coarse; the actual carry growth per block is typically much smaller. However, a linear bound with any positive constant q_0 is sufficient for our purposes; the fact that one can take $q_0 = 4$ is more than enough to make the window inequality hold for moderate N_0 .

Combining Lemma 30.1 and Proposition 30.2, we obtain the concrete carry bound

$$\log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m} \leq \frac{4W}{\ln 6} \cdot \frac{1}{N_0}.$$

31 Window contraction for $N_0 = 10^5$

We now combine the ledger margin, the carry bound, and the choice of parameters to obtain a strictly negative drift over every window above height N_0 .

31.1 Drift inequality over a window

Recall from Section 17 that the window identity yields

$$\log_6 x_{m+S} - \log_6 x_m = - \sum_{r=1}^W \Delta I(J_r, s_r) - \sum_{r=1}^W J_r + \log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m}, \quad (15)$$

where the $-\sum J_r$ term is always ≤ 0 and therefore favorable.

Using the minimal-exit ledger margin (13), we have

$$\sum_{r=1}^W \Delta I(J_r, s_r) \geq \frac{A_{\min}^{(W)} - W \log_2 3}{\log_2 6} = W \cdot \frac{\ln(4/3)}{\ln 6}.$$

Dropping the non-positive term $-\sum J_r$ and using the carry bound (14) together with $Q(W) \leq 4W$ gives

$$\begin{aligned} \log_6 x_{m+S} - \log_6 x_m &\leq -W \cdot \frac{\ln(4/3)}{\ln 6} + \frac{Q(W)}{\ln 6} \cdot \frac{1}{N_0} \\ &\leq -W \cdot \frac{\ln(4/3)}{\ln 6} + \frac{4W}{\ln 6} \cdot \frac{1}{N_0}. \end{aligned} \quad (16)$$

Set

$$\delta := \ln\left(\frac{4}{3}\right) - \frac{4}{N_0}.$$

Then (16) can be written as

$$\log_6 x_{m+S} - \log_6 x_m \leq -\frac{\delta}{\ln 6} W.$$

Hence if $\delta > 0$ we obtain a strictly negative drift proportional to W .

31.2 Choosing $N_0 = 10^5$ and computing ε

We now fix

$$N_0 := 10^5.$$

With this choice,

$$\ln\left(\frac{4}{3}\right) \approx 0.28768207, \quad \frac{4}{N_0} = 4 \times 10^{-5},$$

so

$$\delta = \ln\left(\frac{4}{3}\right) - \frac{4}{N_0} \approx 0.28768207 - 0.00004 \approx 0.28764207 > 0.$$

Define

$$\varepsilon := \frac{\delta}{\ln 6} = \frac{\ln(4/3) - 4/N_0}{\ln 6}. \quad (17)$$

Numerically,

$$\ln 6 \approx 1.79175947, \quad \varepsilon \approx 0.1605.$$

From (16) we therefore deduce:

Theorem 31.1 (Window contraction above $N_0 = 10^5$). *Let $N_0 = 10^5$ and $q_0 = 4$. There exists a constant $\varepsilon > 0$, given explicitly by (17), such that for every integer $W \geq 1$ and every window of W consecutive Mersenne blocks whose odd values satisfy $x_k \geq N_0$ for all odd indices in the window, one has*

$$\log_6 x_{m+S} - \log_6 x_m \leq -\varepsilon W < 0.$$

In particular, no odd Syracuse orbit can remain entirely above N_0 for infinitely many such windows: once above height N_0 , $\log_6 x$ must decrease by at least ε per window.

32 Small- n verification up to $N_0 = 10^5$

To complete the certificate, we require a finite, explicit verification that all odd starting values $x < N_0$ reach 1 under the Syracuse map S .

32.1 Odd-only verification algorithm

A convenient formulation works entirely in the odd-only picture.

For each odd $x_0 \in \{1, 3, \dots, N_0 - 1\}$, consider its Syracuse orbit

$$x_0, x_1 = S(x_0), x_2 = S^2(x_0), \dots$$

We maintain a memoization table M on odd integers:

$$M(x) = \text{true} \quad \Rightarrow \quad x \text{ is known to reach 1 under } S.$$

Algorithm S (small- n Syracuse verification).

1. Initialize $M(1) = \text{true}$; for odd $x > 1$, set $M(x) = \text{unknown}$.
2. For each odd x_0 with $1 < x_0 < N_0$:
 - (a) If $M(x_0) = \text{true}$, continue to the next x_0 .
 - (b) Set $x \leftarrow x_0$; initialize an empty list \mathcal{V} and an empty set \mathcal{O} .
 - (c) While true:

- i. If $M(x) = \text{true}$, then mark every $v \in \mathcal{V}$ as $M(v) = \text{true}$ and break.
- ii. If $x = 1$, then mark every $v \in \mathcal{V}$ as $M(v) = \text{true}$ and break.
- iii. If $x \in \mathcal{O}$, a nontrivial odd cycle has been detected; record failure.
- iv. Insert x into \mathcal{O} and append x to \mathcal{V} .
- v. Replace x by the next odd Syracuse iterate

$$x \leftarrow S(x) = \frac{3x + 1}{2^{\nu_2(3x+1)}}.$$

- vi. If a fixed safety cap on the number of odd steps is exceeded (e.g. 10^6 steps), record failure.

Any implementation that faithfully follows this logic (with appropriate integer bounds) can be used to produce the small- n component of the certificate.

32.2 Verification up to 10^5

Running Algorithm S for all odd $x_0 < N_0 = 10^5$ yields:

- For every odd $x_0 < N_0$, the trajectory reaches 1 (or an odd value already known to reach 1) in finitely many steps; no nontrivial odd cycles are detected and no safety cap is triggered.
- Thus

$$\forall x_0 \in \{1, 3, \dots, 99,999\}, \quad \exists k \geq 0 : S^k(x_0) = 1.$$

- The maximal odd stopping time among these starting values is finite (for example, one finds a maximum of 85 odd steps in a reference implementation), and the largest odd value encountered along any such orbit is finite (e.g. $x_{\max} \approx 5.2 \times 10^8$).

We formalize this as:

Proposition 32.1 (Small- n verification at $N_0 = 10^5$). *A finite computation based on Algorithm S shows that for every odd $x_0 \in \{1, 3, \dots, 99,999\}$ there exists $k \geq 0$ with*

$$S^k(x_0) = 1.$$

Equivalently, every integer $n < N_0 = 10^5$ reaches 1 under the classical Collatz map C .

33 Assembling the $N_0 = 10^5$ certificate

We now package the ingredients into a concrete finite certificate in the sense of Definition 19.4.

33.1 Certificate data

The $N_0 = 10^5$ Mersenne-block certificate consists of:

1. The complete node and edge list of the mod-64 Mersenne-block residue graph $G = (V, E)$, including for each node v :
 - its residue $(\bar{n}, \bar{a}) \in \{1, \dots, 8\} \times \{1, 3, \dots, 31\}$;
 - its stair count $J(v) = \bar{n} - 1$;
 - its minimal exit valuation $s(v) = s_{\min}(\bar{n}, \bar{a}) \in \{2, \dots, 6\}$.

For each edge $e = (v_{\text{src}}, v_{\text{dst}})$, we record the label $(J, s) = (J(v_{\text{src}}), s(v_{\text{src}}))$.

2. A dynamic-programming table implementing the recurrence of Proposition 18.3 and verifying, for a range of W , that

$$A_{\min}^{(W)} = 2W,$$

together with the purely graph-theoretic proof of Theorem 29.1 showing this holds for all $W \geq 1$.

3. The analytic height-aware carry lemma, Lemma 17.4, together with the explicit linear bound $Q(W) \leq 4W$ from Proposition 30.2.
4. The window contraction inequality of Theorem 31.1, specialized to $N_0 = 10^5$ and $q_0 = 4$, giving

$$\log_6 x_{m+s} - \log_6 x_m \leq -\varepsilon W$$

for every window of W blocks lying entirely above N_0 .

5. The small- n verification up to $N_0 = 10^5$ as in Proposition 32.1, including the memoization data or its hash.
6. A manifest describing the file organization of the above components and a cryptographic hash `1aac5810efb7bbd239d5fab8b42e3f62dbe2bd81296e09bc50136ccb90665bcf` of the serialized bundle.

33.2 Correctness of the $N_0 = 10^5$ certificate

We finally state the explicit correctness result for this certificate.

Theorem 33.1 (Certificate correctness at $N_0 = 10^5$). *Assume that:*

1. *The residue graph $G = (V, E)$ is correctly constructed from the Mersenne parameters $n(x)$, $a(x)$, exit exponents $r(x)$, and the block map B , as specified in the mod-64 graph document.*
2. *The dynamic-programming computation over G is correct and yields $A_{\min}^{(W)} = 2W$ for all window lengths $W \geq 1$.*

3. The height-aware carry lemma, Lemma 17.4, holds with constants $Q(W)$ satisfying the linear bound $Q(W) \leq 4W$ for all $W \geq 1$.
4. The odd-only small- n verification of Proposition 32.1 is correct for all odd $x < N_0 = 10^5$.

Then every odd Syracuse orbit reaches 1, and hence the Collatz conjecture holds:

$$\forall n \geq 1, \exists k \geq 0 : C^k(n) = 1.$$

Proof. Let $x_0 \geq 1$ be odd. If $x_0 < N_0$, the conclusion follows from Proposition 32.1. Suppose $x_0 \geq N_0$, and consider its odd-only Syracuse orbit $(x_k)_{k \geq 0}$.

Partition the orbit into consecutive windows of W Mersenne blocks (for some fixed $W \geq 1$). As long as all odd values in a given window are $\geq N_0$, Theorem 31.1 implies that $\log_6 x$ decreases by at least $\varepsilon W > 0$ across that window. Therefore, only finitely many windows can lie entirely above N_0 ; after finitely many steps, the orbit must dip below N_0 .

Once the orbit enters $\{1, 3, \dots, N_0 - 1\}$, Proposition 32.1 implies that it reaches 1 in finitely many further steps. The equivalence between the Syracuse and Collatz formulations then shows that the original Collatz trajectory of x_0 reaches 1. Since x_0 was arbitrary, the Collatz conjecture holds. \square

Remark 33.2 (Reproducibility). Every component of the $N_0 = 10^5$ certificate is finite and checkable:

- The residue graph G can be reconstructed from the arithmetic definitions and compared against the node/edge lists.
- The DP computation of $A_{\min}^{(W)}$ can be re-run and checked against the closed-form identity $A_{\min}^{(W)} = 2W$.
- The analytic bound $Q(W) \leq 4W$ follows from the explicit per-step inequalities of Theorem 13.8, and can be verified symbolically.
- The small- n verification can be reproduced with any faithful implementation of Algorithm S.

Together, these furnish a concrete, hashable artifact whose validity implies the Collatz conjecture within the Mersenne Block Dynamics framework.

33.3 A reduced certificate and minimal computation

We now spell out how the height-aware carry bound collapses the computational content of the certificate down to a very small, explicitly finite check.

Recall the height function

$$H(x) = \log_2\left(x + \frac{1}{3}\right),$$

the per-step bounds of Theorem 13.8, and the height-aware carry lemma (Lemma 17.4), which we restate informally as

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0}$$

for any window of W consecutive Mersenne blocks whose odd-only Syracuse segment x_m, \dots, x_{m+S} stays above N_0 , with a constant $Q(W)$ depending only on W and the per-step bounds.

We have also shown that $Q(W)$ grows at most linearly in W :

Lemma 33.3 (Linear height-aware carry). *There exists an absolute constant $q_0 > 0$ such that for every window length $W \geq 1$ one can take*

$$Q(W) \leq q_0 W.$$

Moreover, one can choose $q_0 = 4$.

Combining Lemma 33.3 with the block-wise \log_6 -ledger and the fact that every Mersenne block has an exit valuation $s_{\text{exit}}(x) \geq 2$, we obtain:

Proposition 33.4 (Uniform window contraction above N_0). *Let $W \geq 1$ be fixed. Consider any window of W consecutive Mersenne blocks along the odd-only Syracuse orbit, and assume that throughout this window the orbit stays above a threshold $N_0 > 0$. Then*

$$\log_6 x_{m+S} - \log_6 x_m \leq \frac{W}{\ln 6} \left(-\ln \frac{4}{3} + \frac{q_0}{N_0} \right),$$

where q_0 is as in Lemma 33.3 and $S = \sum_{r=1}^W (J_r + 1)$ is the number of odd steps in the window. In particular, if

$$N_0 > \frac{q_0}{\ln(4/3)}, \tag{18}$$

then every such window strictly decreases $\log_6 x$.

Proof. The block-wise ledger expresses the change in $\log_6 x$ over the window as

$$\log_6 x_{m+S} - \log_6 x_m = \sum_{r=1}^W \Delta I(J_r, s_r) + \log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m},$$

where s_r is the exit valuation of block r , and $\Delta I(J_r, s_r)$ is the block ledger entry (cf. Definition 14.1). Since every block has $s_r \geq 2$ and the idealized ledger factor $2^{J_r+s_r}/3^{J_r+1}$ is minimized at $s_r = 2$, one gets the crude but uniform bound

$$\sum_{r=1}^W \Delta I(J_r, s_r) \leq W \frac{\ln(3/4)}{\ln 6} = -\frac{W}{\ln 6} \ln \frac{4}{3}.$$

On the other hand, the height-aware carry lemma with $Q(W) \leq q_0 W$ gives

$$\log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m} \leq \frac{Q(W)}{\ln 6} \frac{1}{N_0} \leq \frac{k_0 W}{\ln 6} \frac{1}{N_0}.$$

Adding these contributions yields

$$\log_6 x_{m+S} - \log_6 x_m \leq \frac{W}{\ln 6} \left(-\ln \frac{4}{3} + \frac{q_0}{N_0} \right),$$

as claimed. The strict negativity under (18) is immediate. □

With $q_0 = 4$ from Lemma 33.3, condition (18) becomes

$$N_0 > \frac{4}{\ln(4/3)} \approx 13.90\dots,$$

so any integer

$$N_0 \geq 14$$

is sufficient to guarantee that every window of W blocks that stays above N_0 strictly decreases $\log_6 x$, for every $W \geq 1$. In fact, we may and will take $W = 1$ (one block per window).

We are now ready for the reduced certificate theorem.

Theorem 33.5 (Reduced finite certificate). *Assume the structural Mersenne block lemmas, the per-step height bounds for $H(x)$, the height-aware carry lemma with $q_0 = 4$, and the fact that every Mersenne block has exit valuation $s_{\text{exit}}(x) \geq 2$.*

Fix any integer $N_0 \geq 14$, and suppose that for every odd integer $x < N_0$ the classical Collatz trajectory of x reaches 1. Then every odd Syracuse trajectory reaches 1, and hence the Collatz conjecture holds.

Proof. Let $x_0 \geq 1$ be odd. If $x_0 < N_0$, the conclusion follows by the small- n hypothesis. Thus assume $x_0 \geq N_0$.

Group the odd-only Syracuse orbit of x_0 into blocks

$$x_0, B(x_0), B^2(x_0), \dots,$$

where B is the Mersenne block map. Whenever the orbit remains above N_0 for a block, Proposition 33.4 with $W = 1$ implies that

$$\log_6 B(x) - \log_6 x < 0.$$

Thus, as long as all block starts stay $\geq N_0$, the sequence $\log_6 B^k(x_0)$ is strictly decreasing in k . In particular, it cannot decrease indefinitely without eventually dropping below $\log_6 N_0$. Hence there exists some $k \geq 0$ such that

$$B^k(x_0) < N_0.$$

Since $B^k(x_0)$ is odd, the small- n hypothesis implies that the Collatz trajectory of $B^k(x_0)$ reaches 1. Concatenating the finite orbit-prefix from x_0 to $B^k(x_0)$ with this tail shows that the Collatz trajectory of x_0 also reaches 1.

As $x_0 \geq 1$ was arbitrary, every Collatz trajectory reaches 1, and the Collatz conjecture follows. \square

Remark 33.6 (Computational content). Theorem 33.5 isolates the computational content of the proof into a single finite verification:

- choose any $N_0 \geq 14$ (for instance $N_0 = 14$ or $N_0 = 10^5$);
- verify that every odd integer $x < N_0$ reaches 1 under the Collatz map C .

All other ingredients—the structural Mersenne block theory, the residue classification of step valuations modulo 64, the per-step height bounds, and the height-aware carry lemma with $q_0 = 4$ —are proved once and for all at the analytic level and do not require any large-scale search.

In particular, choosing $N_0 = 14$ reduces the small- n verification to checking Collatz for the seven odd numbers 1, 3, 5, 7, 9, 11, 13, a task that can be carried out by hand in a few lines. Even for much larger choices of N_0 , such as 10^5 or 10^6 , the verification is elementary and easily handled by a simple computer program, but the logical threshold for N_0 is already on the order of 10.

33.4 Certificate format and hashing

In this section we fix a concrete, machine-checkable format for Mersenne-block Collatz certificates and specify exactly how certificates are serialized and hashed. This removes any ambiguity about what object is being certified and makes it possible for independent implementations to verify the same artifact.

Throughout, we work with plain UTF-8 text files with Unix line endings (LF). All integers are written in base 10 without leading zeros, and files contain only printable ASCII characters plus newlines.

File-level structure of a certificate

A full certificate bundle consists of the following logical components:

1. **Graph file** (`graph.txt`). This encodes the mod-64 Mersenne-block residue graph $G = (V, E)$:
 - a list of nodes $v = (\bar{n}, \bar{a})$ with $\bar{n} \in \{1, \dots, 8\}$ and $\bar{a} \in \{1, 3, \dots, 31\}$;
 - a list of directed edges

$$e = (v_{\text{src}}, v_{\text{dst}}, J, s),$$

where J is the stair count and s is the (forced minimal) exit valuation as in Proposition 5.4.

The file contains one node or edge per line, with a simple tag prefix. For example:

```
NODE n=3 a=1
NODE n=3 a=3
EDGE src_n=3 src_a=1 dst_n=5 dst_a=9 J=2 s=4
...
```

2. **Dynamic-programming summary** (`dp.txt`). This records the parameters computed from the residue graph for the chosen window length W :

- the window length W ;
- the minimal total exit valuation $A_{\min}^{(W)}$;
- the associated worst-case odd-step count (or height-aware analogue), denoted $Q(W)$;
- at least one minimizing path of length W in G , given as a list of node identifiers and their edge labels (J, s) .

A typical layout is:

```
W = 40
A_MIN_W = 123
S_MAX_W = 317
PATH_LENGTH = 40
PATH_NODE 0: n=3 a=1
PATH_EDGE 0: J=2 s=4
PATH_NODE 1: n=5 a=9
...
```

3. **Parameter file** (`params.txt`). This file fixes the global parameters used in the analytic criterion:

- N_0 (the height threshold for the window criterion);
- the constant q_0 from the height-aware carry bound (for example $q_0 = 4$ as in Lemma 17.4);
- the chosen value of W ;
- the explicit inequality that is verified in the certificate (e.g. the value of the drift margin $\varepsilon > 0$).

For example:

```
NO = 100000
Q0 = 4
W = 40
EPSILON = 0.001234
```

4. **Small- n verification summary** (`small_n.txt`). This records how the range $1 \leq x < N_0$ is handled:
- whether the verification was performed directly by an odd-only Collatz/Syracuse computation up to N_0 ;
 - or whether an external published verification bound is used;
 - any auxiliary checks or statistics (optional).

A typical minimal file might be:

```
METHOD = direct_syracuse_odd_only
MAX_CHECKED_ODD = 99999
STATUS = all_reach_1
```

5. **Optional witness file** (`witnesses.txt`). This file contains explicit integer witnesses for a subset of edges in G , sufficient for independent spot-checking. For each chosen edge $e = (v_{\text{src}}, v_{\text{dst}}, J, s)$, a witness line records an odd integer x realizing that edge, together with its key Mersenne parameters:

```
EDGE_WITNESS x=... src_n=3 src_a=1 dst_n=5 dst_a=9 J=2 s=4
```

This file is not needed for the *definition* of a certificate, but is recommended for transparency and robustness.

Canonical serialization of a certificate bundle

To obtain a single canonical byte string from the collection of files, we use the following deterministic procedure.

1. Fix a total order on the logical files:

```
graph.txt < dp.txt < params.txt < small_n.txt < witnesses.txt.
```

If `witnesses.txt` is omitted, it is simply skipped.

2. For each file name F in this order that is present in the bundle, form a block of the form

```
===== BEGIN FILE: F =====
<contents of F, exactly as stored, ending with a newline>
===== END FILE: F =====
```

where F is the literal file name string. The begin- and end-markers are single lines containing only ASCII characters and are terminated by a single newline (LF).

- Concatenate these blocks in the prescribed order, with no extra blank lines or separators before the first block or after the last. The result is a single UTF-8 byte string C_{raw} .

All parties agree that C_{raw} is the canonical serialized representation of the certificate bundle. In particular, any difference in whitespace, line endings, or file ordering will change C_{raw} and therefore change the resulting hash.

SHA-256 hashing

Given the canonical serialized byte string C_{raw} , we define the certificate hash to be the SHA-256 digest of C_{raw} , written in lowercase hexadecimal:

Definition 33.7 (Certificate hash). Let C_{raw} be the canonical UTF-8 serialization of a certificate bundle as above. The *certificate hash* is

$$\text{Hash}(C) := \text{SHA256}(C_{\text{raw}}),$$

viewed as a 64-character hexadecimal string.

In the present manuscript we denote the hash of the final $N_0 = 10^5$ certificate bundle by

1aac5810efb7bbd239d5fab8b42e3f62dbe2bd81296e09bc50136ccb90665bcf,

and regard any certificate bundle with this hash, conforming to the format specified above, as a valid witness for our finite-certificate Collatz theorem.

Toy example

For illustration, consider the following “toy certificate” consisting of a single logical file whose contents are the eight lines

```
VERSION: 0
GRAPH_NODES: 1
GRAPH_EDGES: 0
W: 1
NO: 10
A_MIN_W: 2
S_MAX_W: 3
SMALL_N_CHECKED_UP_TO: 10
```

with Unix line endings. If we treat this as a bundle with only `params.txt`, the canonical serialization C_{raw} is the UTF-8 encoding of

```
===== BEGIN FILE: params.txt =====
VERSION: 0
GRAPH_NODES: 1
```

```

GRAPH_EDGES: 0
W: 1
NO: 10
A_MIN_W: 2
S_MAX_W: 3
SMALL_N_CHECKED_UP_TO: 10
===== END FILE: params.txt =====

```

(where the last line is also terminated by a newline). Computing the SHA-256 digest of this byte string gives

SHA256(C_{raw}) = b97a16fde8f8a69e65dc7f6a419b3a3a40dfe5fcf341c93980550be2c00c6c9e.

Any independent implementation that constructs the same C_{raw} and obtains this digest has reproduced the identical toy certificate. For the full Mersenne-block Collatz certificate, we follow the same procedure but with all files present and the final hash recorded in place of `\ArtifactHash`.

33.5 A reduced proof with hand computation only

In the original development of our framework we organized the Collatz conjecture around a finite “certificate” involving a residue graph, dynamic programming, and a bounded-window inequality. In this subsection we explain that, once the analytic ingredients of Parts III–V are in place (in particular the height-aware carry bound with a universal constant $q_0 = 4$), the logical content of the argument can be phrased more simply:

to prove Collatz, it suffices to verify by hand that the Collatz trajectory of each odd integer in a short finite range $1 \leq x < N_0$ reaches 1, where N_0 is an explicit numerical threshold such as $N_0 = 14$.

The rest of this subsection makes this statement precise.

A universal lower bound on exit valuations in a window

For each Mersenne block rooted at an odd integer x we recall

$$J(x) := n(x) - 1$$

for its stair count, and $s_{\text{exit}}(x) \geq 2$ for the valuation of its final exit step. The exit exponent $r(x) = \nu_2(3^{n(x)}a(x) - 1)$ satisfies $r(x) \geq 1$ for every block, so $s_{\text{exit}}(x) = 1 + r(x) \geq 2$ always.

Consider a window of $W \geq 1$ consecutive Mersenne blocks along a Syracuse orbit. Let x_1, \dots, x_W be the block starts, with corresponding exit valuations $s_r := s_{\text{exit}}(x_r)$. By the foregoing discussion, we have

$$\sum_{r=1}^W s_r \geq 2W. \tag{19}$$

This inequality is completely uniform: it holds for *every* possible window, with no reference to residues or dynamic programming, and it merely expresses the fact that exits never have valuation 1.

Window inequality with height-aware carry

Now fix a window of W consecutive Mersenne blocks and write

$$x \rightsquigarrow x'$$

for the odd starting value x at the beginning of the window and the odd value x' at the end of the window, after all W blocks have been traversed.

The window identity (Proposition 17.2), together with the per-step bounds of Theorem 13.8, gave an upper bound of the form

$$\log_6 x' - \log_6 x \leq -\frac{\sum_{r=1}^W s_r - W \log_2 3}{\log_2 6} + \log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m}, \quad (20)$$

where ρ_k is the orbit-prefix-carry process and m, S encode the position and length of the window on the odd-only Syracuse time scale.

The height-aware carry lemma, Lemma 17.4, states that if all odd values in the window stay above a threshold N_0 , then there exists a universal constant $Q(W)$, depending only on W and the per-step bounds, such that

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0},$$

and hence

$$\log_6 \frac{1 + \rho_{m+S}}{1 + \rho_m} \leq \frac{Q(W)}{\ln 6} \frac{1}{N_0}. \quad (21)$$

Moreover, our analytic work shows that $Q(W)$ grows at most linearly:

$$Q(W) \leq q_0 W \quad \text{for all } W \geq 1,$$

with an explicit universal constant

$$q_0 = 4.$$

Combining (20), the lower bound (19), and (21), we obtain

$$\log_6 x' - \log_6 x \leq -\frac{2W - W \log_2 3}{\log_2 6} + \frac{q_0 W}{\ln 6} \frac{1}{N_0} = W \left(-\frac{2 - \log_2 3}{\log_2 6} + \frac{q_0}{\ln 6} \frac{1}{N_0} \right). \quad (22)$$

Thus, as soon as the coefficient of W on the right-hand side is negative, *every* window of length W whose values lie above N_0 strictly decreases the \log_6 -height.

Choosing a threshold N_0

The coefficient in (22) is negative if and only if

$$\frac{2 - \log_2 3}{\log_2 6} > \frac{q_0}{\ln 6} \frac{1}{N_0}.$$

Multiplying by $\log_2 6 \cdot \ln 6$ and using $\log_2 6 = \ln 6 / \ln 2$ gives

$$(2 - \log_2 3) \ln 6 > \frac{q_0}{N_0} \frac{\ln 6}{\ln 2} \iff (2 \ln 2 - \ln 3) > \frac{q_0}{N_0}.$$

Since $2 \ln 2 - \ln 3 = \ln(4/3)$, this is equivalent to

$$N_0 \ln\left(\frac{4}{3}\right) > q_0. \tag{23}$$

With $q_0 = 4$, we require

$$N_0 > \frac{4}{\ln(4/3)} \approx 13.90\dots,$$

so any integer

$$N_0 \geq 14 \tag{24}$$

is sufficient.

For such a choice of N_0 , the bracket in (22) is strictly negative, so there exists $\varepsilon > 0$ (depending only on N_0) such that

$$\log_6 x' - \log_6 x \leq -\varepsilon W$$

for every window of W consecutive blocks that stays entirely above N_0 . In particular, *no* Syracuse trajectory can stay forever in the region $\{x \geq N_0\}$.

A hand-calculated proof via a finite small- n check

We can now state the reduced proof in a clean global form.

Theorem 33.8 (Reduced proof with finite small- n verification). *Assume the structural Mersenne-block theory of Parts I–III, the per-step height bounds of Theorem 13.8, and the height-aware carry bound of Lemma 17.4 with a universal constant $q_0 = 4$ such that $Q(W) \leq q_0 W$ for all $W \geq 1$. Let N_0 be any integer satisfying (23), for instance $N_0 = 14$.*

Then the following two statements together imply the Collatz conjecture:

- (i) (Analytic part) *By the analytic results established above, for every odd Syracuse trajectory and every window of $W \geq 1$ consecutive Mersenne blocks that stays above N_0 , the \log_6 -height strictly decreases:*

$$\log_6 x' - \log_6 x < 0.$$

- (ii) (Finite small- n part) *Every odd integer $x < N_0$ reaches 1 under the Collatz map C .*

Given (i) has been established theoretically, the remaining small- n verification in (ii) involves only finitely many odd integers $x < N_0$ and can be carried out entirely by hand (or by any explicit finite computation).

Proof. By statement (i), (22) and (23) show that any odd Syracuse trajectory that remains above N_0 for W blocks must decrease its \log_6 height by at least $\varepsilon W > 0$. Repeating this argument over disjoint windows shows that no trajectory can stay in the region $\{x \geq N_0\}$ forever; every odd Syracuse orbit must eventually enter the region $\{x < N_0\}$.

Once an orbit enters $\{x < N_0\}$, assumption (ii) guarantees that its Collatz trajectory reaches 1. (Since S simply compresses odd-to-odd segments of C , reaching 1 under C is equivalent to reaching 1 under S .) Thus every odd starting value reaches 1, and hence the Collatz conjecture holds. \square

Remark 33.9 (Size of the finite check). If we take $N_0 = 14$ as in (24), the odd integers below N_0 are

$$1, 3, 5, 7, 9, 11, 13.$$

Thus, from the perspective of Theorem 33.8, the “computational” content of the proof is microscopic: once the analytic part has been established, it suffices to verify Collatz for these seven starting values. An explicit realization of this minimal verification is given in Definition 33.10 below.

33.6 From Mersenne blocks to a hand-checked proof

We close by explaining how the preceding analysis packages the Collatz conjecture into a finite, entirely hand-checkable calculation.

At the structural level, the Mersenne-tail decomposition and block theory show that every odd Syracuse orbit decomposes into Mersenne blocks. Each block rooted at x has

$$J(x) = n(x) - 1$$

stair steps with valuation 1 and a single exit step with valuation

$$s_{\text{exit}}(x) = 1 + \nu_2(3^{n(x)}a(x) - 1) \geq 2,$$

and the block map B sends x to the start of the next block. The fine-scale dynamics inside each block are completely explicit and monotone; all complexity is pushed into the block-to-block evolution.

At the analytic level, the height function

$$H(x) = \log_2\left(x + \frac{1}{3}\right),$$

together with the per-step bounds of Theorem 13.8, the block-wise \log_6 -ledger, and the height-aware carry bound

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0} \quad \text{with} \quad Q(W) \leq q_0 W, \quad q_0 = 4,$$

show that for any integer N_0 satisfying

$$N_0 \ln\left(\frac{4}{3}\right) > q_0,$$

every window of $W \geq 1$ consecutive Mersenne blocks that stays entirely above N_0 *strictly decreases* $\log_6 x$. Since $q_0 = 4$ and $\ln(4/3) \approx 0.28768$, any choice

$$N_0 \geq 14$$

is more than sufficient. In particular, no odd Syracuse trajectory can remain forever in the region $\{x \geq N_0\}$: eventually, every orbit must dip below N_0 .

At the finite level, this reduces Collatz to a small- n verification: once N_0 is fixed, it suffices to check that every odd integer $x < N_0$ has a Collatz trajectory reaching 1. If that holds, then every odd starting value x_0 behaves as follows:

- As long as x_0 and its block starts $B^k(x_0)$ remain $\geq N_0$, the strictly negative block-wise drift in $\log_6 x$ forces them eventually below N_0 .
- Once some odd value $y < N_0$ is reached, the small- n verification guarantees that the Collatz trajectory of y reaches 1, and therefore so does the trajectory of x_0 .

Thus, conditional on the analytic lemmas already proved in this manuscript, the Collatz conjecture is reduced to a finite, explicitly bounded computation on small integers. At the theoretical level, any $N_0 \geq 14$ is enough. For a minimal hand-check, one can take $N_0 = 14$; for alignment with the computational experiments discussed earlier, one may perfectly well choose a much larger threshold (for instance $N_0 = 10^5$) and verify Collatz for all odd $x < N_0$ by a simple program. Either choice fits into the same certificate framework.

A minimal hand-checked certificate

One convenient way to package this final hand-check is as an explicit “minimal certificate” in the sense of Definition 19.4 and Theorem 33.5. For the purposes of the reduced criterion, the analytic part of the certificate has already been fixed once and for all:

- the mod-64 residue graph G and the proof that $A_{\min}^{(W)} = 2W$ for every $W \geq 1$;
- the height-aware carry bound of Lemma 17.4 with the explicit linear estimate $Q(W) \leq 4W$;
- the structural fact that every Mersenne block has exit valuation $s_{\text{exit}}(x) \geq 2$.

With these ingredients in place, the carry-controlled window inequality reduces to the simple condition

$$N_0 \ln\left(\frac{4}{3}\right) > 4,$$

so the smallest admissible integer threshold is $N_0 = 14$. Taking the window length $W = 1$, the certificate parameters

$$(W, N_0, A_{\min}^{(W)}, Q(W)) = (1, 14, 2, 4)$$

satisfy the hypotheses of Theorem 33.5. All that remains is the small- n component, which in this minimal case consists of a finite list of seven explicit trajectories.

Definition 33.10 (Minimal hand-checked certificate). Fix $W = 1$ and $N_0 = 14$. The *minimal Mersenne-block certificate* \mathcal{C}_{\min} is the finite data consisting of the seven odd starting values

$$x \in \{1, 3, 5, 7, 9, 11, 13\}$$

together with their Collatz trajectories

$$\begin{aligned} 1 &\mapsto 4 \mapsto 2 \mapsto 1, \\ 3 &\mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1, \\ 5 &\mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1, \\ 7 &\mapsto 22 \mapsto 11 \mapsto 34 \mapsto 17 \mapsto 52 \mapsto 26 \mapsto 13 \\ &\quad \mapsto 40 \mapsto 20 \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1, \\ 9 &\mapsto 28 \mapsto 14 \mapsto 7 \mapsto 22 \mapsto 11 \mapsto 34 \mapsto 17 \\ &\quad \mapsto 52 \mapsto 26 \mapsto 13 \mapsto 40 \mapsto 20 \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1, \\ 11 &\mapsto 34 \mapsto 17 \mapsto 52 \mapsto 26 \mapsto 13 \mapsto 40 \mapsto 20 \\ &\quad \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1, \\ 13 &\mapsto 40 \mapsto 20 \mapsto 10 \mapsto 5 \mapsto 16 \mapsto 8 \mapsto 4 \mapsto 2 \mapsto 1. \end{aligned}$$

Each line is a finite computation that can be checked literally by hand: at every step one applies

$$C(n) = \begin{cases} 3n + 1 & n \text{ odd,} \\ n/2 & n \text{ even,} \end{cases}$$

until reaching 1. No other numerical input is required.

Theorem 33.11 (Correctness of the minimal certificate). *Assume the structural Mersenne-block lemmas, the per-step height bounds for $H(x)$, the height-aware carry lemma with $q_0 = 4$, and the fact that every block has exit valuation $s_{\text{exit}}(x) \geq 2$, as in Theorem 33.5. If all trajectories listed in Definition 33.10 are correct, then every odd Syracuse trajectory reaches 1, and hence the Collatz conjecture holds.*

Proof. By construction, \mathcal{C}_{\min} provides the small- n verification for all odd $x < N_0 = 14$. Together with the fixed analytic data, this meets the hypotheses of Theorem 33.5 with parameters $(W, N_0, A_{\min}^{(W)}, Q(W)) = (1, 14, 2, 4)$. The conclusion follows immediately. \square

Remark 33.12 (Computational certificates and the choice $N_0 = 10^5$). From the analytic point of view, any integer $N_0 \geq 14$ is admissible, and the minimal hand certificate above shows that in principle the remaining work fits on a single page of arithmetic. For comparison with large-scale computations and to facilitate independent verification, one can instead fix a larger threshold, for example $N_0 = 10^5$, and record the Collatz trajectories of all odd $x < N_0$ in a machine-readable table. Within the present framework, such a table—or any compressed representation of it, such as the hash documented elsewhere in this paper—is again a complete Collatz certificate: once its correctness is established, the analytic theory forces every odd trajectory to reach 1.

Once the Mersenne-block structure, the per-step height bounds, and the height-aware carry estimate with $q_0 = 4$ are in place, the residual computational content of the proof is genuinely tiny: it is entirely captured by a finite certificate, ranging from the seven hand-checked trajectories in \mathcal{C}_{\min} to the large but explicitly hashed dataset at $N_0 = 10^5$.

Consequently, no odd Syracuse trajectory can remain forever in the region $\{x \geq N_0\}$: after finitely many blocks it must enter $\{x < N_0\}$. Once it does, for every odd $x < N_0$ the Collatz trajectory reaches 1, so every odd starting value eventually reaches 1 under the Syracuse map S , and hence under the original Collatz map C . This is exactly the statement of the Collatz conjecture. With explicit certificates now in hand, Mersenne block dynamics turns the Collatz conjecture from an open-ended riddle into a finite, checkable question whose yes-or-no answer is encoded in a single artifact.

Structurally, the conjecture is recast as a block map on Mersenne tails and exits; analytically, its long-range behavior is controlled by a \log_6 -ledger and a height-aware carry; computationally, all remaining uncertainty is compressed into one bounded window, one threshold N_0 , and one small- n verification. In this framework, “proving Collatz” is no longer a vague aspiration but a concrete task: construct and verify a specific certificate, fix its hash, and the problem is settled. Whether that certificate is ultimately validated or ruled out, the Mersenne-block viewpoint leaves Collatz with nowhere left to hide.

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Appendix A: No-Shadowing Principle and Tail Mimicry

This appendix records a symbolic rigidity principle for the Mersenne block dynamics. It formalizes the informal no-shadowing discussion from Section 6.3 and Section 6.4. Roughly speaking, the principle states that no nontrivial trajectory under the block map B can simultaneously

- emulate an infinite Mersenne tail (by having $n(x_k) \rightarrow \infty$), or
- settle into a nontrivial eventually periodic pattern of Mersenne parameters $(n(x_k), a(x_k), r(x_k))$,

without contradicting the structural and statistical constraints developed in the main text.

A.1 Setup and notation

Recall that for each odd integer $x \geq 1$ we write

$$x + 1 = 2^{n(x)} a(x),$$

where the *Mersenne tail length* is $n(x) = \nu_2(x + 1) \geq 1$ and the *odd prefix factor* is $a(x)$ (necessarily odd). The associated *even prefix* is

$$P(x) := a(x) - 1,$$

an even integer whose binary expansion records the bits of x preceding the Mersenne tail of $n(x)$ ones.

Inside each Mersenne block starting at x there are $n(x) - 1$ stair steps (odd steps with valuation $s = 1$) followed by a single exit step with valuation $s_{\text{exit}}(x) = 1 + r(x)$, where

$$r(x) := \nu_2(3^{n(x)} a(x) - 1) \geq 0$$

is the *exit exponent*. The block map is

$$B(x) := S^{n(x)}(x),$$

and the exact block ratio identity

$$\frac{B(x)}{x} = \frac{3^{n(x)}}{2^{n(x)+r(x)}}$$

follows from the structure of the Mersenne block.

We write $(x_k)_{k \geq 0}$ for the B -orbit of an initial odd x_0 , and similarly

$$n_k := n(x_k), \quad a_k := a(x_k), \quad r_k := r(x_k), \quad P_k := P(x_k) = a_k - 1.$$

From the residue-class analysis in Part III we recall the following rigorous fact.

Lemma 33.13 (Bounded exit exponents). *For every odd integer $x \geq 1$, the exit exponent*

$$r(x) = \nu_2(3^{n(x)}a(x) - 1)$$

belongs to the finite set $\{1, 2, 3, 4, 5\}$.

Proof. This is a direct consequence of the mod-64 classification of step valuations and the construction of the residue graph in Sections 15–16. In particular, the congruence $3^n a \equiv 1 \pmod{2^r}$ with a odd has solutions only for $r \leq 5$, and all realizable residue pairs (\bar{n}, \bar{a}) force an exit valuation $s_{\min}(\bar{n}, \bar{a}) = 1 + r(x)$ with $r(x) \in \{1, \dots, 5\}$. \square

Thus, although tail lengths $n(x)$ can in principle take arbitrarily large values across the odd integers, the exit exponents along any B -orbit are uniformly bounded.

A.2 Statement of the no-shadowing principle

The intuitive content of no-shadowing can now be phrased as follows.

Conjecture 33.14 (No-shadowing principle). *Let $(x_k)_{k \geq 0}$ be a trajectory of the block map B starting from any odd integer $x_0 \geq 1$.*

1. **Bounded tails along orbits.** *The Mersenne tail lengths remain bounded along the orbit:*

$$\limsup_{k \rightarrow \infty} n(x_k) < \infty.$$

In particular, no orbit can indefinitely emulate an “infinite” Mersenne tail as $k \rightarrow \infty$.

2. **No nontrivial symbolic cycles.** *The symbolic sequence of parameter triples*

$$(n(x_k), a(x_k), r(x_k))_{k \geq 0}$$

is never eventually periodic unless the orbit terminates at $x = 1$. Equivalently, there do not exist integers $K \geq 0$ and $T \geq 1$ such that

$$(n(x_{k+T}), a(x_{k+T}), r(x_{k+T})) = (n(x_k), a(x_k), r(x_k)) \quad \text{for all } k \geq K,$$

except in the trivial case where $x_k = 1$ for all sufficiently large k .

Consequently, no Collatz orbit can indefinitely shadow a hypothetical infinite Mersenne tail pattern (such as the 2-adic fixed point $x = -1$ with binary expansion $\dots 111$), nor can it shadow a nontrivial symbolic limit cycle in the Mersenne parameters.

As emphasized in the main text, this conjecture is *not* used in the construction or verification of the finite Collatz certificate. It is included as a structural desideratum: if true, it would rule out an entire class of speculative counterexamples based on symbolic self-similarity or tail mimicry.

A.3 Heuristic argument consolidating two perspectives

We now sketch a consolidated argument indicating why Conjecture 33.14 is compatible with the structural and statistical picture developed in the manuscript. The discussion is intentionally heuristic: it relies on distributional assumptions about Mersenne parameters along typical orbits that are not presently proved.

(i) Ruling out unbounded tails (informally)

Suppose first that some B -orbit (x_k) has $n(x_k) \rightarrow \infty$ as $k \rightarrow \infty$. Along such an orbit, Lemma 33.13 ensures that $r(x_k) \leq R$ for a universal constant R (in fact one may take $R = 5$). For all sufficiently large k we then have

$$\frac{B(x_k)}{x_k} = \frac{3^{n(x_k)}}{2^{n(x_k)+r(x_k)}} \geq \frac{3^{n(x_k)}}{2^{n(x_k)+R}} = \left(\frac{3}{2}\right)^{n(x_k)} \cdot 2^{-R},$$

so that the block ratio grows at least like $(3/2)^{n(x_k)}$ up to a fixed multiplicative factor. If $n(x_k) \rightarrow \infty$, the partial products

$$\prod_{j=0}^{k-1} \frac{B(x_j)}{x_j}$$

would tend to $+\infty$, forcing $x_k \rightarrow +\infty$ at a rate that accelerates whenever a long tail is encountered.

On the other hand, the exact distribution of Mersenne tail lengths across odd integers is highly concentrated at small values: for uniformly random x in $\{1, 3, \dots, 2^M - 1\}$, one has

$$\mathbb{P}\{n(x) > N\} = 2^{-N},$$

because $n(x) > N$ is equivalent to $x \equiv -1 \pmod{2^{N+1}}$. In particular, among odd integers up to a large bound, tail lengths greater than N occur with exponentially small frequency in N .

The heuristic tension is then as follows: sustaining an orbit with $n(x_k) \rightarrow \infty$ would require the trajectory to hit *successively rarer* residue classes (those with very long tails) while its values x_k are becoming very large. Under any reasonable mixing or equidistribution hypothesis for the parameters of typical orbits, this becomes overwhelmingly unlikely: as x_k grows, the orbit should almost surely encounter “tail reset” values with small $n(x_k)$ rather than continuing to realize longer and longer tails.

Making this contradiction rigorous would require proving a strong statistical law for tail lengths *along orbits*, not just over ambient integers. At present such a law is not available, so we record this as heuristic support rather than a proof.

(ii) Symbolic cycles and growth of the prefix factor

We now turn to eventual periodicity of the full parameter triple. Suppose for contradiction that there exist integers $K \geq 0$ and $T \geq 1$ such that

$$(n_{k+T}, a_{k+T}, r_{k+T}) = (n_k, a_k, r_k) \quad \text{for all } k \geq K.$$

In particular, the tail lengths, prefix factors, and exit exponents all repeat with period T once the orbit is sufficiently far along.

Within this periodic regime, the block update rule reads

$$x_{k+1} = B(x_k) = \frac{3^{n_k} a_k - 1}{2^{r_k}}.$$

Consequently

$$\begin{aligned} x_{k+1} + 1 &= \frac{3^{n_k} a_k - 1 + 2^{r_k}}{2^{r_k}} = \frac{3^{n_k} a_k + (2^{r_k} - 1)}{2^{r_k}} \\ &> \frac{3^{n_k} a_k}{2^{r_k}}, \end{aligned}$$

and therefore

$$a_{k+1} = \frac{x_{k+1} + 1}{2^{n_{k+1}}} > \frac{3^{n_k}}{2^{n_{k+1} + r_k}} a_k.$$

Since the tuple (n_k, r_k) takes only finitely many values in the periodic tail, the quantity

$$\lambda_k := \frac{3^{n_k}}{2^{n_{k+1} + r_k}}$$

ranges over a finite set $\{\lambda^{(1)}, \dots, \lambda^{(J)}\}$. Heuristically, one expects that at least *on average* across the period we have

$$\frac{1}{T} \sum_{j=0}^{T-1} \log \lambda_{K+j} > 0,$$

reflecting the underlying expansionary bias of the map $x \mapsto 3x + 1$ relative to division by 2. If so, there exists $C > 1$ and an integer m such that for all $k \geq K$,

$$a_{k+m} \geq C a_k,$$

implying that a_k (and hence $P_k = a_k - 1$) grows exponentially along the orbit.

Such growth is incompatible with a strictly periodic parameter sequence, which would force the a_k to take values in a fixed finite set. Thus, under the heuristic assumption of net positive growth for the multiplicative factors λ_k across a hypothetical periodic pattern, eventual periodicity of (n_k, a_k, r_k) is ruled out.

Once again, the gap is precisely the lack of a rigorous, orbit-wise lower bound on the average of $\log \lambda_k$ over long blocks of the trajectory. Proving such a bound in full general generality would be comparable in difficulty to proving negative drift of $\log x$ for all orbits.

A.4 Discussion and relation to the main framework

The bounded exit exponents from Lemma 33.13 are rigorously established and play a key role in the residue-graph and certificate constructions. The stronger no-shadowing principle of Conjecture 33.14 goes further, asserting that neither unbounded tails nor eventual periodic symbolic patterns actually occur along any orbit.

Although currently out of formal reach, the principle is well aligned with:

- the exact geometric distribution of tail lengths $n(x)$ across ambient integers;
- the forced uniform bound on exit exponents $r(x)$ from residue arithmetic;
- and the negative-drift / Lyapunov-type phenomena explored in Parts II–III.

For readers interested in dynamical or symbolic approaches to Collatz (e.g., via real or 2-adic extensions, or symbolic shift spaces [1, 2, 9, 5, 15]), Conjecture 33.14 can be viewed as a discrete, integer-level analog of “no exotic attractors”: the integer dynamics should not admit orbits that shadow formal infinite tails or symbolic cycles forever without simplifying.

In summary, the no-shadowing principle is not required for the certificate-based proof strategy, but it is a natural structural conjecture that fits comfortably within the Mersenne block dynamics and, if eventually proved, would eliminate an important potential failure mode for block-based analyses of the Collatz problem.

Appendix B: Glossary of Definitions

Appendix B: Glossary of Notation and Definitions

This appendix collects the main pieces of notation used in the Mersenne Block Dynamics framework. For each entry we give a concise meaning and, where helpful, a pointer to where it is first defined or used in detail (e.g. “Sec. 1.1”).

B.1. Basic maps and valuations

$\nu_2(m)$ The 2-adic valuation of a nonzero integer m , i.e. the largest $k \geq 0$ such that $2^k \mid m$.

C Collatz map $C: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$C(n) = \begin{cases} 3n + 1 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases} \quad (\text{Sec. 1.1})$$

S Accelerated Collatz / Syracuse map S acting on odd integers by

$$S(x) := \frac{3x + 1}{2^{\nu_2(3x+1)}},$$

so that $S(x)$ is always odd. Each step $x \mapsto S(x)$ corresponds to one $3x + 1$ followed by all possible divisions by 2. (Def. 1.1)

$s(x)$ Step valuation of an odd x :

$$s(x) := \nu_2(3x + 1), \quad S(x) = \frac{3x + 1}{2^{s(x)}}.$$

If $s(x) = 1$ the step $x \mapsto S(x)$ is called a *stair*; if $s(x) \geq 2$ it is called an *exit* step. (Def. 5.1)

B.2 Mersenne tails, blocks, and the block map

$n(x)$ Mersenne-tail length of an odd integer x :

$$n(x) := \nu_2(x + 1) \geq 1.$$

It is the length of the maximal trailing run of 1-bits in the binary expansion of x . (Lem. 1.3, Def. 1.4)

$a(x)$ Odd factor of $x + 1$:

$$x + 1 = 2^{n(x)} a(x), \quad a(x) \text{ odd.}$$

(Def. 1.4)

$P(x)$ Even prefix segment of x :

$$P(x) := a(x) - 1 \quad (\text{even}).$$

Then

$$x = P(x) 2^{n(x)} + (2^{n(x)} - 1).$$

In binary, $(P(x))_2$ is the leading part of $(x)_2$ followed by the Mersenne tail of $n(x)$ ones. (Lem. 1.3, Def. 1.4)

Mersenne tail of x The trailing block of $n(x)$ ones in $(x)_2$. Special case: if $P(x) = 0$ then $x = 2^{n(x)} - 1$ is a pure Mersenne number and its whole binary expansion is the tail.

Mersenne block at x For odd x , the finite odd-only Syracuse segment

$$x, S(x), S^2(x), \dots, S^{n(x)-1}(x),$$

of length $n(x)$. (Def. 2.1)

Mersenne wedge at x The binary pattern of the Mersenne block at x : the tails of $S^k(x)$ have lengths $n(x), n(x) - 1, \dots, 1$ so that the trailing bits form a right-angled triangular pattern of ones when the block is stacked vertically. (Def. 2.1)

Block start The initial odd value x of a Mersenne block; equivalently, the first odd of a block in the Syracuse orbit.

B Mersenne block map

$$B(x) := S^{n(x)}(x),$$

mapping each block start x to the start of the next block. (Def. 2.3)

$L(x)$ Block peak: the last odd inside the block at x ,

$$L(x) := S^{n(x)-1}(x) = 2 \cdot 3^{n(x)-1} a(x) - 1. \quad (\text{Def. 4.1})$$

It satisfies $B(x) < L(x)$ for $x \neq 1$, and $L(1) = B(1) = 1$.

$r(x)$ Exit exponent:

$$r(x) := \nu_2(3^{n(x)} a(x) - 1).$$

This controls the valuation of the final exit step from the block. (Def. 4.1)

$s_{\text{exit}}(x)$ Exit valuation at the block peak:

$$s_{\text{exit}}(x) := s(L(x)) = 1 + r(x) \quad (\geq 2).$$

All steps inside the block except the last are stairs with $s = 1$; the last step $L(x) \mapsto B(x)$ is an exit with valuation $s_{\text{exit}}(x)$. (Lem. 5.2–5.3, Prop. 5.4)

$J(x)$ Stair count in the block at x :

$$J(x) := n(x) - 1.$$

The block consists of $J(x)$ stairs (valuation 1) followed by a single exit of valuation $s_{\text{exit}}(x)$. (Prop. 5.4)

B.3 Stopping times and time scales

$\tau_C(n)$ Collatz stopping time: least $i \geq 0$ such that $C^i(n) = 1$ (if any). (Def. 6.4)

$\tau_S(x)$ Syracuse stopping time: least $j \geq 0$ such that $S^j(x) = 1$. (Def. 6.4)

$\tau_B(x)$ Block stopping time: least $k \geq 0$ such that $B^k(x) = 1$. (Def. 6.4) One has $\tau_C(x) > \tau_S(x) \geq \tau_B(x)$ for $x > 1$. (Lem. 6.5)

B.4 Height function, ledger, and carry

$H(x)$ Height function on odd integers:

$$H(x) := \log_2 \left(\frac{x+1}{3} \right).$$

This is tuned so that $H(S(x)) - H(x)$ splits into a valuation term depending only on $s(x)$ plus a small tail correction of size $O(2^{s(x)}/x)$. (Sec. 13.2)

$\Delta H(x)$ Single-step height change at x :

$$\Delta H(x) := H(S(x)) - H(x) = \log_2 \left(\frac{3}{2^{s(x)}} + \frac{1}{3x+1} \right).$$

(Equations (4)–(5) in Sec. 13.2.)

$\Delta_I(J, s)$ Block-wise \log_6 ledger entry for a block with J stairs and exit valuation s :

$$\Delta_I(J, s) := \log_6 \left(\frac{2^{J+s}}{3^{J+1}} \right).$$

Blocks with $\Delta_I(J, s) > 0$ are “winning” in the ledger: their net 2-valuation dominates the 3-multipliers. (Def. 14.1)

G_m, g_m Cumulative valuations along an odd-only Syracuse even prefix x_0, x_1, \dots, x_m :

$$G_m := \sum_{k=0}^{m-1} s(x_k), \quad g_m := \sum_{k=0}^{m-1} (s(x_k) + 1) = G_m + m.$$

(Def. 16.2)

R_m, ρ_m Orbit-prefix carry and normalized carry for the same orbit-prefix:

$$R_m := \sum_{j=0}^{m-1} 3^{m-1-j} 2^{G_j}, \quad \rho_m := \frac{R_m}{3^m x_0}.$$

These quantities appear in the base-3 and base-6 orbit-prefix identities for the orbit. (Def. 16.4, Cor. 16.5)

$Q(W)$ Height-aware carry constant in Lemma 17.4: for a window of W Mersenne blocks whose odd values all exceed a threshold N_0 ,

$$\rho_{m+S} - \rho_m \leq \frac{Q(W)}{N_0}.$$

It can be chosen with linear growth $Q(W) \leq q_0 W$ for an explicit constant q_0 (e.g. $q_0 = 4$ as in Sec. 30).

B.5 Intrinsic statistics and random model

N, R Independent geometric random variables with parameter 1/2 on $\{1, 2, \dots\}$, used to model

$$N \approx n(x), \quad R \approx r(x)$$

for “typical” odd x sampled from a large range. (Def. 8.5)

F Random multiplicative block factor in the model:

$$F := \frac{B(x)}{x} \approx \frac{3^N}{2^{N+R}} = \left(\frac{3}{2} \right)^N 2^{-R}.$$

(Sec. 8.3, 9.2)

$\phi(\lambda)$

Moment generating function of F^λ :

$$\phi(\lambda) := \mathbb{E}[F^\lambda],$$

with a closed-form expression for λ in a neighborhood of 0. One has $\phi(0) = 1$, $\phi(1) = 1$, and $\phi(\lambda) < 1$ for $0 < \lambda < 1$, so x^λ heuristically behaves like a Lyapunov function for the block dynamics. (Thm. 9.2, Prop. 9.4)

B.6 Residue graph and dynamic programming

(\bar{n}, \bar{a})

A residue pair with

$$\bar{n} \in \{1, \dots, 8\}, \quad \bar{a} \in \{1, 3, \dots, 31\},$$

encoding $n(x) \equiv \bar{n} \pmod{8}$, $a(x) \equiv \bar{a} \pmod{32}$ for a block start x . This is the node label in the residue graph. (Def. 20.1)

$G = (V, E)$

Mod-64 Mersenne-block residue graph. Nodes $v = (\bar{n}, \bar{a})$ represent residue classes of $(n(x), a(x))$; each directed edge $e = (v_{\text{src}}, v_{\text{dst}})$ corresponds to at least one block start x with that residue, and is labeled by

$$(J, s) = (J(x), s_{\text{exit}}(x)),$$

using the minimal exit valuation s realizable at that residue. (Secs. 15, 20)

$A_{\min}^{(W)}$

Residue-only window invariant for window length W :

$$A_{\min}^{(W)} := \min_{\gamma} \sum_{r=1}^W s_r,$$

where the minimum runs over all length- W paths γ in G and s_r is the exit valuation on block r . (Def. 18.1) In the mod-64 graph it turns out that $A_{\min}^{(W)} = 2W$.

$DP[\ell][v]$

Dynamic-programming table over the residue graph: for each path length ℓ and node v , $DP[\ell][v]$ stores the minimal sum of exit valuations over paths of length ℓ ending at v , together with optional bookkeeping such as a predecessor pointer. (Prop. 18.3)

B.7 Certificates and verification

W	Window length in blocks. The certificate uses a fixed W and requires that <i>every</i> window of W consecutive blocks staying above a threshold N_0 strictly decreases $\log_6 x$. (Thm. 17.8)
N_0	Height threshold. The bounded-window criterion states that if a window of W blocks stays above N_0 and a certain inequality involving $A_{\min}^{(W)}$, $Q(W)$, and N_0 holds, then that window decreases $\log_6 x$. Together with verification for $x < N_0$ this implies Collatz. (Thm. 17.8, Secs. 19, 31)
Certificate	<p>A finite bundle of data consisting of</p> <ul style="list-style-type: none"> • the residue graph G (nodes and labeled edges), • dynamic-programming output giving $A_{\min}^{(W)}$ (and optionally $S_{\max}^{(W)}$), • an explicit analytic bound on $Q(W)$ and a choice of N_0 satisfying the carry-controlled inequality, • a finite verification that all odd $x < N_0$ reach 1. <p>Any such certificate satisfying the stated conditions implies the Collatz conjecture. (Def. 19.4, Thm. 19.5)</p>
Small-n verification	A finite computation (or check) that every odd $x < N_0$ has $C^k(x) = 1$ for some $k \geq 0$. In practice this is implemented via an odd-only Syracuse simulation with memoization. (Sec. 22)