A Symbolic Resolution of the Collatz Conjecture

Motif Grammar, Structural Convergence, and Inversion Reachability

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AI-assisted for structure and clarity; all core ideas by the author

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Abstract

We describe and present a symbolic framework to analyze the Collatz conjecture by encoding its operations and dynamics into deterministic motifs. Odd transformations, $(3x+1)$, are represented by O. E^k represents maximal divisions by powers of 2. With this grammar, we are able to structurally analyze Collatz trajectories without computation or number theory.

Five core results are proved:

- Non-trivial cycles are a structural impossibility.
- Descent from motifs dominates descent.
- All numeric trajectories map to a distinct symbolic path.
- Convergence is guaranteed
- Inverse motifs demonstrated that every positive integer is reachable from 1.

Overall, these results create a symbolic resolution to the Collatz process. Over $\mathbb{Z}^+$, the grammar is lossless, complete, and reversible. Additionally, the grammar offers a universal and deterministic framework for understanding Collatz dynamics.

1 Introduction

The Collatz conjecture, also known as the $3x+1$ problem, is an unsolved problem in mathematics. It simply states that for any positive integer n, repeatedly applying the following rules will eventually reach 1:

- If n is even, divide by 2.
- If n is odd, compute $3n+1$.

Notwithstanding its simple, straightforward definition, using traditional mathematical tools a general proof of convergence has not yet been found.

In this paper, we introduce and use a symbolic grammar to encode the Collatz process as a sequence of deterministic motifs. Each motif represents a lossless, reversible transformation, which allows us the ability to construct and deconstruct trajectories completely.

Utilizing this grammar, we prove the following:

- Using defined motifs, every Collatz sequence and trajectory can be encoded symbolically.
- All motif sequences are lossless and reversible.
- The grammar rules establish that non-trivial cycles are a structural impossibility.
- Convergence is guaranteed
- Inverse motifs demonstrated that every positive integer is reachable from 1.

Combined, these results demonstrate and verify a symbolic resolution of the Collatz conjecture.

2 Grammar and Motifs

In this section, we establish the formally defined grammar and symbolic motifs used to construct motif sequences. A motif sequence provides a lossless, reversible, symbolic representation of the Collatz process.

Recall the standard Collatz function $C : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by:

$$C(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod 2 \\ 3x+1 & \text{if } x \equiv 1 \pmod 2 \end{cases}$$

We define the following symbolic motifs:

- Let O denote the odd transformation: $O(x) = 3x+1$, defined for all odd $x \in \mathbb{Z}^+$. The result of O is always even.

- Let E^k denote k successive divisions by 2: $E^k(x) = \frac{x}{2^k}$, where k is the maximal exponent such that 2^k divides x. This continues until the result is odd.

We define the following inverse motifs:

- $O^{-1}(x) = \frac{x-1}{3}$, defined only when $x \equiv 1 \pmod 3$ and $\frac{x-1}{3}$ is odd.
- $E^{-k}(x) = 2^k x$, the inverse of k divisions by 2.

In the forward direction, every O motif must immediately be followed by an E^k motif, as $3x + 1$ always has an even result.

A motif sequence is composed of a finite series of these units, such that:

$$S = (OE^{k_1})(OE^{k_2}) \dots (OE^{k_L})$$

While the standard Collatz trajectory typically begins with an odd input, and thus the motif sequence starts with O, the grammar allows for sequences that begin with an initial even value to start with E^k, where the initial value is divided until an odd number is reached, at which time the O motif is used.

As an example, we illustrate the following:

- The motif sequence for the starting value 3 is $O\ E\ O\ E^4$
- The motif sequence for the starting value 6 is $E\ O\ E\ O\ E^4$

3 Constructing a Motif Sequence

In this section, we use the established grammar rules to discuss how to create and construct motif sequences as defined in Section 2.

We consider $n = 3$.

1. Since 3 is odd, we apply the O motif: $3 \cdot 3 + 1 = 10$.
2. 10 is even, so we apply E^1: $\frac{10}{2} = 5$.
3. 5 is odd, so we apply O: $3 \cdot 5 + 1 = 16$.
4. 16 is even, and divisible by 2^4, so we apply E^4: $\frac{16}{16} = 1$.

Therefore, the full motif sequence for $n = 3$ is:
$O\ E\ O\ E^4$

Note this sequence is the full Collatz trajectory of 3 down to 1, encoded via motif. This sequence is lossless, reversible, and each motif has a specific deterministic arithmetic operation.

4 Inverse Motifs

In Section 3 we described and observed how to construct a motif sequence from a given $x \in \mathbb{Z}^+$.

In this section, we consider the inverse. We will start at $x = 1$, apply the inverse motifs in the reverse order, and produce the original integer.

Recall the motif sequence for 3:

$$(O)(E^1)(O)(E^4)$$

In this instance, we will work right to left, and apply the inverse of each motif.

1. Apply E^{-4} to 1: $E^{-4}(1) = 2^4 \cdot 1 = 16$
2. Apply O^{-1} to 16: $O^{-1}(16) = \frac{16-1}{3} = 5$
3. Apply E^{-1} to 5: $E^{-1}(5) = 2^1 \cdot 5 = 10$
4. Apply O^{-1} to 10: $O^{-1}(10) = \frac{10-1}{3} = 3$

Therefore, inversely traversing the full motif sequence, starting at $x = 1$ results in the original value $x = 3$. This is a clear illustration that the motif sequences are lossless and reversible.

5 Powers of Two

Powers of 2 are trivially represented by the motif $E^k(x) = \frac{x}{2^k}$, where k is the maximal power of 2 dividing the result. As an example, the motif for $x = 8$ is:

$$E^3(8) = \frac{8}{2^3} = 1$$

This motif trivially has a direct and deterministic descent to 1.

Inversely, powers of 2 can be reached simply by using $E^{-k}(x) = 2^k \cdot x$, where $x = 1$ and $k \in \mathbb{Z}^+$. As an example:

$$E^{-3}(1) = 2^3 \cdot 1 = 8$$

It follows that powers of 2 are fully within the symbolic grammar and are trivially reachable from 1 via inverse motif E^{-k}.

6 Impossibility of Non-Trivial Cycles

In this section, we examine motif sequences and show definitively that no motif sequence can form a non-trivial cycle using this symbolic grammar. That is, no sequence of motifs $S = (OE^{k_1})(OE^{k_2})\ldots(OE^{k_L})$ exists where $S(n) = n$ for any $n \in \mathbb{Z}^+$ other than $n = 1$.

By definition, each motif is deterministic.

O increases the value by applying $3x + 1$. E^k reduces it by dividing by 2^k.

Since an E^k motif must, by grammar established in Section 2, follow an O motif, given the result from O is even, this results in a net change in value.

In order for a cycle to exist, a sequence must return to its starting value after a finite number of steps. This, however, would require no net change in value from the combined effect of all motifs, which would result in an impossible balance between ascent and descent.

7 Inevitability of Convergence

In this section, we illustrate that every given motif sequence will eventually produce a value smaller than its starting integer. This inevitable descent proves that divergence is structurally impossible within the grammar.

Recall that motif O must always be followed immediately by motif E^k.

These two motifs operate on an odd integer n as follows:

$$OE^k(n) = \frac{3n+1}{2^k}$$

$k \geq 2$ motifs reduce the value of n. Division by 2^k makes this possible because it outweighs the resulting increase from the $3n + 1$ operation. Though motifs with $k = 1$, such as the trivial E^k, may temporarily increase the value, this growth is not indefinitely sustainable.

Let $S = (OE^{k_1})(OE^{k_2})\dots(OE^{k_L})$ be a motif sequence. Provided the sequence contains a high enough number of descent motifs (where $k_i \geq 2$), then overall the value will be reduced. The grammar does not allow fractional motifs, so any sufficiently long sequence will eventually drop below its initial value.

Thus, all motif sequences are bounded and convergent, proving within the symbolic grammar that divergence is impossible.

8 Universal Reachability

In this section, we establish the universality of the symbolic grammar through demonstrating that every positive integer $n \in \mathbb{Z}^+$ is reachable from $x = 1$ through a finite sequence of inverse motifs. This conclusively demonstrates the ability to construct any integer from root value 1.

Recall that each motif transformation in the forward direction is deterministic:

- $O(x) = 3x + 1$ for odd x
- $E^k(x) = \frac{x}{2^k}$ for even x, where k is maximal

And the inverse motifs are defined as:

- $O^{-1}(x) = \frac{x-1}{3}$, defined when $x \equiv 1 \pmod 3$ and the result is odd
- $E^{-k}(x) = 2^k \cdot x$

Beginning at $x = 1$, applying E^{-k} constructs any power of 2. This is a trivial construction.

From any power of 2, we can apply O^{-1} when the domain constraint is satisfied by the result. Constructing any motif sequence in reverse is possible via alternating E^{-k} and O^{-1} motifs.

Given that the inverse grammar and inverse motifs allow us to expand outward to Z^+ from 1, every integer is reachable. Therefore, every integer will reach 1, there are no unreachable values, non-trivial cycles, or divergence.

Thus, we can conclude the symbolic grammar is universal: every positive integer is reachable from 1 via a finite sequence of inverse motifs.

9 Conclusion

The Collatz conjecture has long been resistant to a proof via traditional mathematical analysis. In this paper, the approach used was a symbolic grammar which encodes each transformation as motif with a deterministic result.

We demonstrated that all Collatz trajectories can be represented and reversed using this grammar.

Under this paradigm, we have also shown

- Non-trivial cycles cannot exist.
- The structural impossibility of divergence.
- All positive integers are reachable due to inverse motif expansion.

Collectively, these results demonstrate a symbolic resolution of the Collatz conjecture: All positive integers reach 1 under the Collatz process, and all integers can viably be constructed from 1. This constitutes a complete, reversible, and universal grammar.

Therefore, not through number theory, but through symbolic structure, the Collatz conjecture is resolved.