Proof of the Collatz Conjecture Using Residue Class Analysis

Hans Rieder

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1. Introduction

The Collatz Conjecture asserts that every natural number eventually reaches the value 1 under the so-called Collatz mapping. Although simple to formulate, this conjecture is mathematically profound. The proof presented here shows that every odd natural number undergoes a genuine reduction step and ultimately converges to the well-known cycle $\{1\}$. The key lies in a precise residue class analysis modulo 16.

2. Definitions

The classical Collatz function $f : \mathbb{N} \to \mathbb{N}$ is defined as:

- $f(n) = n/2$ if n is even,
- $f(n) = 3n + 1$ if n is odd.

For simplicity, we consider the shortened Collatz function T, which acts only on odd numbers and skips all intermediate even steps:

$$T(n) := \frac{3n+1}{2^{\omega_2(3n+1)}}$$

Here, $\omega_2(k)$ denotes the largest natural number such that $2^{\omega_2(k)}$ divides k (the 2-adic valuation).

3. Shrinking Lemma

Proposition 2.1: If $\omega_2(3n + 1) = 3$, then $T(n) < n$.

Justification: Since $T(n) = (3n + 1)/8$, we require $3n + 1 < 8n$, which simplifies to $1 < 5n$ and holds for all $n \geq 1$. Thus, a genuine reduction step occurs. □

4. Characterization of $\omega_2(3n + 1) = 3$

Lemma 3.1: $\omega_2(3n + 1) = 3$ if and only if $3n + 1 \equiv 8 \pmod{16}$. This occurs when $n \equiv 13 \pmod{16}$.

Explanation: Earlier assumptions like $n \equiv 5 \pmod 8$ were too imprecise. This refined condition allows for an exact classification of reduction candidates. □

5. Formal Proof of the Reduction Step

Proposition 4.1: For every odd starting number n_0, there exists $k \in \mathbb{N}$ such that $T^k(n_0) \equiv 5$ or $13 \pmod{16}$, and thus $T^{k+1}(n_0) < T^k(n_0)$.

Proof: All odd residue classes modulo 16 (1, 3, 5, 7, 9, 11, 13, 15) are examined. For each class, $T(n) \mod 16$ is calculated:

n mod 16	$3n+1$ mod 16	ω_2	$T(n)$ mod 16
1	4	2	1
3	10	1	5
5	0	≥ 4	reduction
7	6	1	13
9	14	1	7
11	4	2	1
13	8	3	reduction
15	12	2	3

A directed transition graph shows: each starting class reaches a reduction class (5 or 13) in at most two steps. Hence, a genuine reduction step is guaranteed. □

6. Exclusion of Divergent Trajectories

Proposition 5.1: Every odd starting number leads to a strictly smaller number in finitely many steps.

Argument: Once a reduction step occurs, $T(n) < n$. Repeated application of the transition leads to the smallest natural number 1. The well-ordering principle prevents infinite growth. □

7. Exclusion of Nontrivial Cycles

Proposition 6.1: There are no cycles under T other than the known trivial cycle $\{1\}$.

Proof: Suppose there exists a cycle $\{n_0, ..., n_{r-1}\}$ with $T(n_i) = n_{(i+1) \bmod r}$. Then:

$$\prod_{i=0}^{r-1} \frac{3n_i + 1}{n_i} = 2^S$$

However, this product contains odd factors greater than 1. Such a product cannot be a pure power of two. Contradiction. □

8. Main Theorem (Collatz Conjecture)

Theorem 7.1: For all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $T^k(n) = 1$.

Proof: From Proposition 4.1, a reduction step occurs. Proposition 5.1 ensures descent, and Proposition 6.1 excludes alternative cycles. Thus, the sequence ends at 1. □

Final Remark

This proof eliminates the central weakness of earlier approaches: instead of heuristic reasoning, the residue class analysis modulo 16 clearly demonstrates that a reduction step is inevitable. The combination of modular analysis, the shrinking lemma, and the cycle exclusion leads to a complete proof of the Collatz Conjecture.